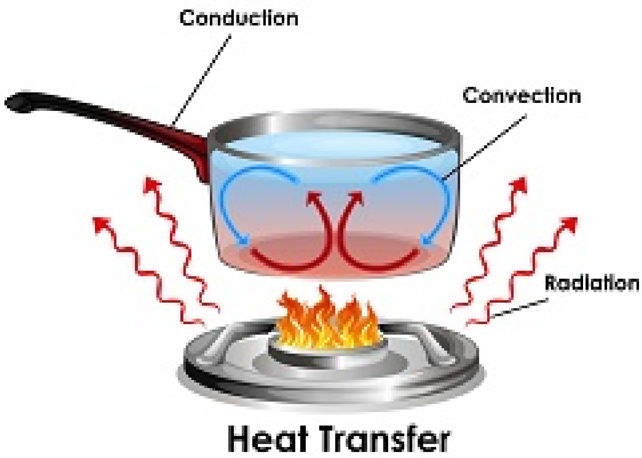
DEPARTMENT OF MECHANICAL ENGINEERING

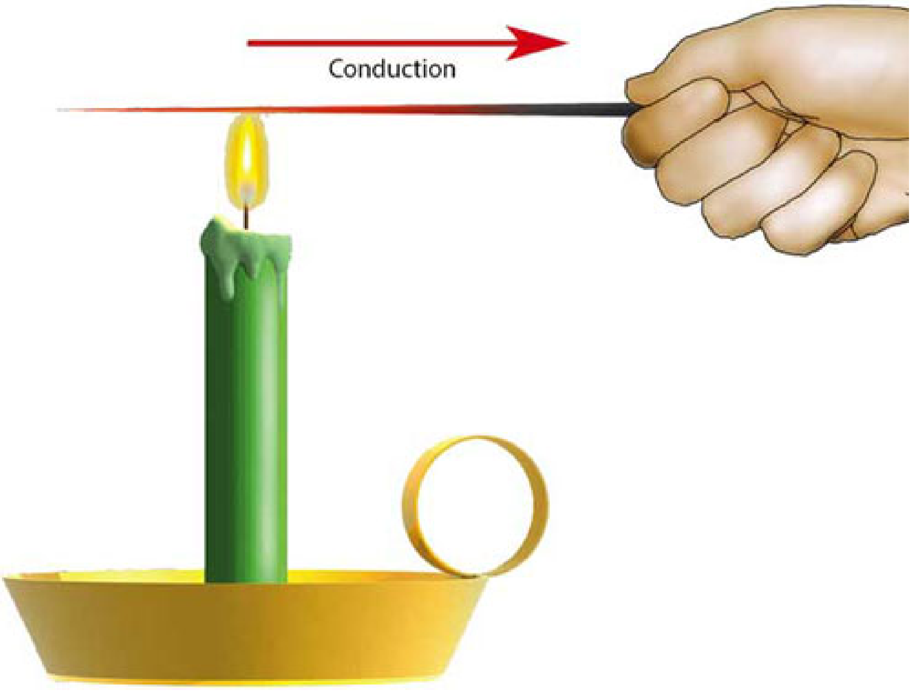
ABIT

HEAT TRANSFER

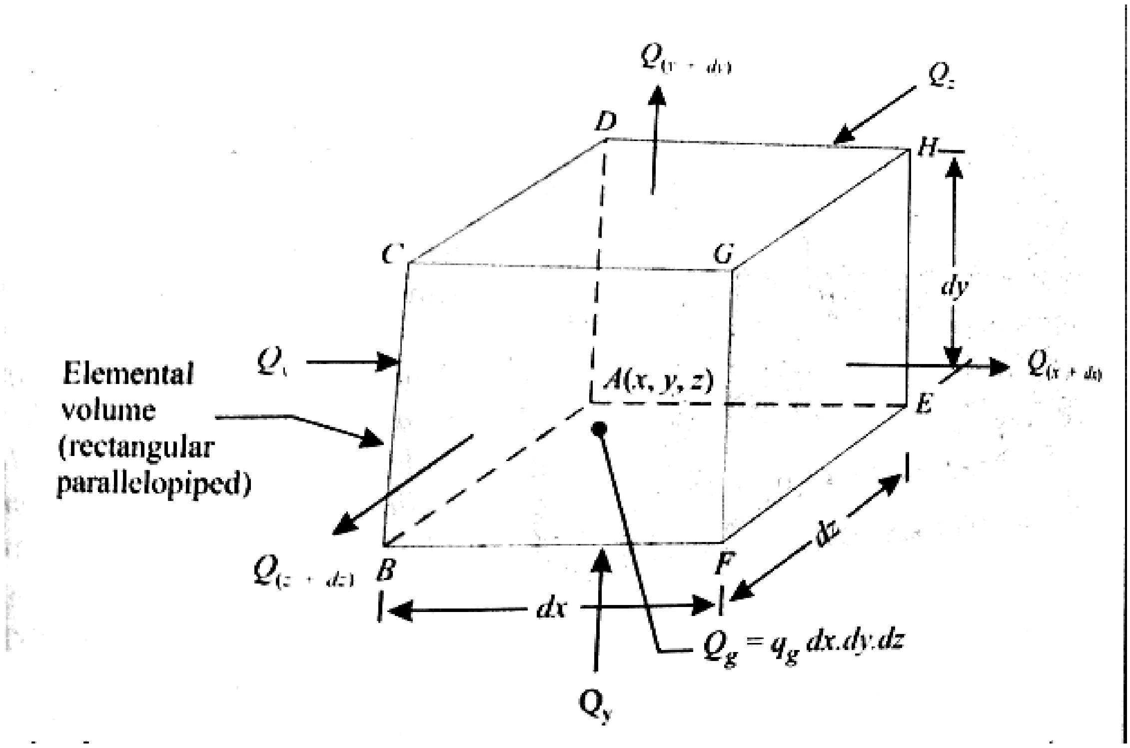


MODULE-I

HEAT CONDUCTION



GENERAL HEAT CONDUCTION EQUATION IN CARTESIAN COORDINATES



Consider an infinitesimal rectangular parallelepiped of sides dx, dy and dz parallel , respectively, to the three axes (X,Y,Z) in a medium in which temperature is varying with location and time as shown in the figure above.

Let, t=temperature at the left face ABCD. This temperature may be assumed uniform over the entire surface, since the area of this face can be made arbitrarily small, and

dt/dx = temperature changes and rate of change along X-direction.

Then, $\left(\frac{\partial t}{\partial x}\right)dx =$ change of temperature through distance dx, and

t+$\left(\frac{\partial t}{\partial x}\right)dx =$ temperature on the right face EFGH (at a distance dx from the left face ABCD).

Further, let, K_X, K_Y, K_Z=Thermal conductivities along X,Y AND Z axes.

If the directional characteristics of a material are same, it is called an “isotropic material” and if different “anisotropic material”.

q_g=Heat generated per unit volume per unit time.

Inside the control volume there may be heat sources due to flow of electric current in electric motors and generators, nuclear fission etc.

ENERGY BALANCE EQUATION FOR VOLUME ELEMENT

Net heat accumulated in the element due to conduction of heat from all the coordinate directions considered (A) + heat generated within the element (B)=Energy stored in the element(C)..........(1)

Let , Q= Rate of heat flow in a direction, and

Q'=(Qdτ)= Total heat flow (flux) in that direction (in time dτ).

A. Net heat accumulated in the element due to conduction of heat from all the directions considered:

Quantity of heat flowing into the element from the left face ABCD during the time interval dτ in X-direction is given by: Heat influx, $Q'_X = -K_X(dy.dz)\frac{\partial t}{\partial x}.d\tau$ ....(i)

During the same time interval dτ the heat flowing out of the right face of control volume(EFGH) will be:

Heat efflux, $Q'_{(x+dx)} = Q'_x + \frac{\partial}{\partial x}(Q'_X)dx$ ....(ii)

Hence heat accumulation in the element due to heat flow in X-direction,

$$dQ'_X = Q'_X - [Q'_X + \frac{\partial}{\partial X}(Q'_X)dx] \qquad \text{[subtracting (ii) from (i)]}$$

$$= -\frac{\partial}{\partial x}(Q'_X)dx$$

$$= -\frac{\partial}{\partial x}[-k_x(dydz)\frac{\partial t}{\partial x}.d\tau]dx$$

$$= \frac{\partial}{\partial x}[k_x\frac{\partial t}{\partial x}]dxdydzd\tau \qquad \text{..........(2.1)}$$

Similarly the heat accumulated due to heat flow by conduction along Y and Z directions in time dτ wil be:

$$dQ'_Y = \frac{\partial}{\partial y}[k_y\frac{\partial t}{\partial y}]dxdydzd\tau \qquad \text{..........(2.2)}$$

$$dQ'_z = \frac{\partial}{\partial z}[k_z\frac{\partial t}{\partial z}]dxdydzd\tau \qquad \text{..........(2.3)}$$

Hence net heat accumulated in the element due to conduction of heat from all the coordinate directions considered

$$= \frac{\partial}{\partial x}[k_x\frac{\partial t}{\partial x}]dxdydzd\tau + \frac{\partial}{\partial y}[k_y\frac{\partial t}{\partial y}]dxdydzd\tau + \frac{\partial}{\partial z}[k_z\frac{\partial t}{\partial z}]dxdydzd\tau =$$

$$[\frac{\partial}{\partial x}(k_x\frac{\partial t}{\partial x}) + \frac{\partial}{\partial y}(k_y\frac{\partial t}{\partial y}) + \frac{\partial}{\partial z}(k_z\frac{\partial t}{\partial z})]dx.dy.dz.d\tau \qquad \text{..........(2.4)}$$

B.Total heat generated within the element(Q'_g):
The total heat generated in the element is given by

$Q'_g = q_g\,(dxdydz)d\tau$ ……………………………………………………………………………………………..(2.5)

C.Energy stored in the element:
The total heat accumulated in the element due to heat flow along coordinate axes (eqn.2.4) and the heat generated within the element (eqn. 2.5) together serve to increase the thermal energy of the element. This increase in thermal energy is given by

$\rho(dxdydz)c.\frac{\partial t}{\partial \tau}.d\tau$ ……………………………………………………………………………………….(2.6)

[since heat stored in the body=mass of the body x specific heat of the body material rise in temperature of body].
Now, substituting eqns. (2.4),(2.5),(2.6), in eqn.(1) , we have

$$\left[\frac{\partial}{\partial x}\left(k_x\frac{\partial t}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial t}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial t}{\partial z}\right)\right]dx.dy.dz.d\tau + q_g\,(dxdydz)d\tau = \rho(dxdydz)c.\frac{\partial t}{\partial \tau}.d\tau$$

Dividing both sides by dx.dy.dz.dτ, we have

$$\left[\frac{\partial}{\partial x}\left(k_x\frac{\partial t}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial t}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial t}{\partial z}\right)\right] + q_g = \rho c.\frac{\partial t}{\partial \tau} \quad \text{…………………………………………………………….(2.7)}$$

Or, using the vector operator ∇ , we get

$$\nabla.(k\nabla t)+q_g=\rho.c.\frac{\partial t}{\partial \tau} \quad \text{……………………………………………………………………………….[2.7(a)]}$$

This is known as the general heat conduction equation for non-homogeneous material, self heat generating and unsteady three dimensional heat flow.

GENEARAL HEAT CONDUCTION EQUATION FOR CONSTANT THERMAL CONDUCTIVITY:

In case of homogeneous(in which properties e.g., specific heat, density, thermal conductivities. are same everywhere in the material) and isotropic(in which properties are independent of surface orientation) material, k_x, k_y. k_z=k and the equation 2.7, becomes

$$\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}+\frac{\partial^2 t}{\partial z^2}+\frac{q_g}{k}=\frac{\rho c}{k}.\frac{\partial t}{\partial \tau}=\frac{1}{\alpha}\frac{\partial t}{\partial \tau} \quad \text{………………………………………………………………………. (2.8)}$$

Where, α=k/ρc =thermal conductivity/ thermal capacity

The quantity, α=k/ρc is known as thermal diffusivity

- The larger the value of α , the faster will the heat diffuse through the material and its temperature will change with time. This will result either due to a high value of thermal conductivity k or a low value of heat capacity ρc. A low value of heat capacity means the less amount of heat entering the element, would be absorbed and used to raise its temperature and more would be available for onward transmission. Metals and gases have relatively high value of α and their response to temperature changes is quite rapid. The non-metallic solids and liquids respond slowly to temperature changes because of their relatively small value of thermal diffusivity.
- Thermal diffusivity is an important characteristic quantity for unsteady conduction situations.

Equation 2.8 by using Laplacian ∇^2 may be written as:

$$\nabla^2 t + \frac{q_g}{k} = \frac{1}{\alpha}.\frac{\partial t}{\partial \tau} \quad \text{.......................................[2.8(a)]}$$

Other simplified forms of heat conduction equation in Cartesian coordinates:

(i) For the case when no internal source of heat generation is present , eqn. 2.8 reduces to

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} = \frac{1}{\alpha}\frac{\partial t}{\partial \tau}$$ [unsteady state ($\left(\frac{\partial t}{\partial \tau} \neq 0\right)$ heat flow with no internal heat generation]

Or, $$\nabla^2 t = \frac{1}{\alpha}.\frac{\partial t}{\partial \tau} \text{ (Fourier's equation) ....(2.9)}$$

(ii) Under the situations when temperature does not depend on time, the conduction then takes place in the steady state(i.e., $\left(\frac{\partial t}{\partial \tau} = 0\right)$) and the equation 2.8 reduces to

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} + \frac{q_g}{k} = 0$$

Or, $$\nabla^2 t + \frac{q_g}{k} = 0 \text{ (Poisson's equation) ....(2.10)}$$

In the absence of internal heat generation, eqn. 2.10 reduces to

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} = 0$$

Or, $$\nabla^2 t = 0 \text{ (Laplace equation) ....(2.11)}$$

GENERAL HEAT CONDUCTION EQUATION IN CARTESIAN COORDINATES-LINK BELOW

https://www.youtube.com/watch?v=BfYG3R3eo0c

HEAT CONDUCTION THROUGH A COMPOSITE WALL

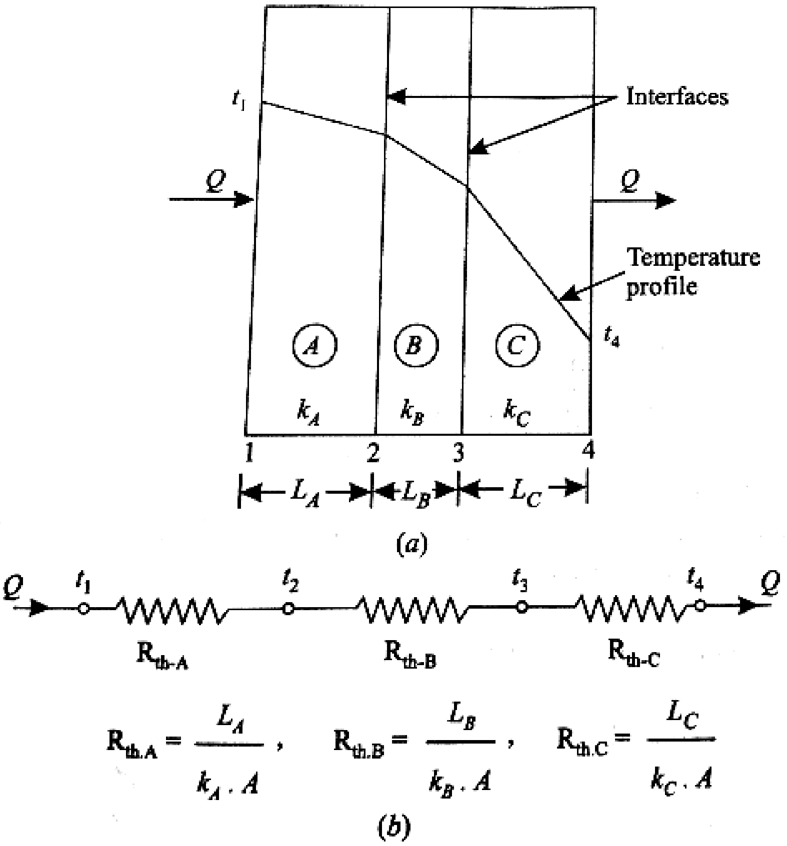


Fig. Steady state conduction through a composite wall.

Consider the transmission of heat through a composite wall consisting of a number of slabs.

Let, L_A, L_B, L_C = Thickness of slabs A, B and C respectively,

K_A, K_B, K_C =Thermal conductivities of slabs A, B and C respectively,

t_1 , t_4 (t_1> t_4)= Temperatures at the wall surfaces 1 and 4 respectively, and

t_2 .t_3 =Temperatures at the interfaces 2 and 3 respectively.

Since the quantity of heat transmitted per unit time through each slab is same, we have,

$$Q=\frac{K_A A(t_1 - t_2)}{L_A} = \frac{K_B A(t_2 - t_3)}{L_B} = \frac{K_C A(t_3 - t_4)}{L_C}$$

(Assuming that there is a perfect contact between the layers and no temperature drop occurs across the interface between the materials)

Rearranging the above expression, we get

$$t_1 - t_2 = \frac{QL_A}{K_A A} \quad \text{...(i)}$$

$$t_2 - t_3 = \frac{QL_B}{K_B A} \quad \text{...(ii)}$$

$$t_3 - t_4 = \frac{QL_C}{K_C A} \quad \text{...(iii)}$$

Adding (i), (ii), and (iii) , we have

$$(t_1 - t_4) = Q\left[\frac{L_A}{K_A A} + \frac{L_B}{K_B A} + \frac{L_C}{K_C A}\right]$$

$$Q = \frac{A(t_1 - t_4)}{\left[\frac{L_A}{K_A} + \frac{L_B}{K_B} + \frac{L_C}{K_C}\right]} \quad \text{...(iv)}$$

Or, $$Q = \frac{(t_1 - t_4)}{\left[\frac{L_A}{K_A A} + \frac{L_B}{K_B A} + \frac{L_C}{K_C A}\right]} = \frac{(t_1 - t_4)}{\left[R_{th-A} + R_{th-B} + R_{th-C}\right]} \quad \text{..(v)}$$

If the composite wall consists of n slabs/layers, then

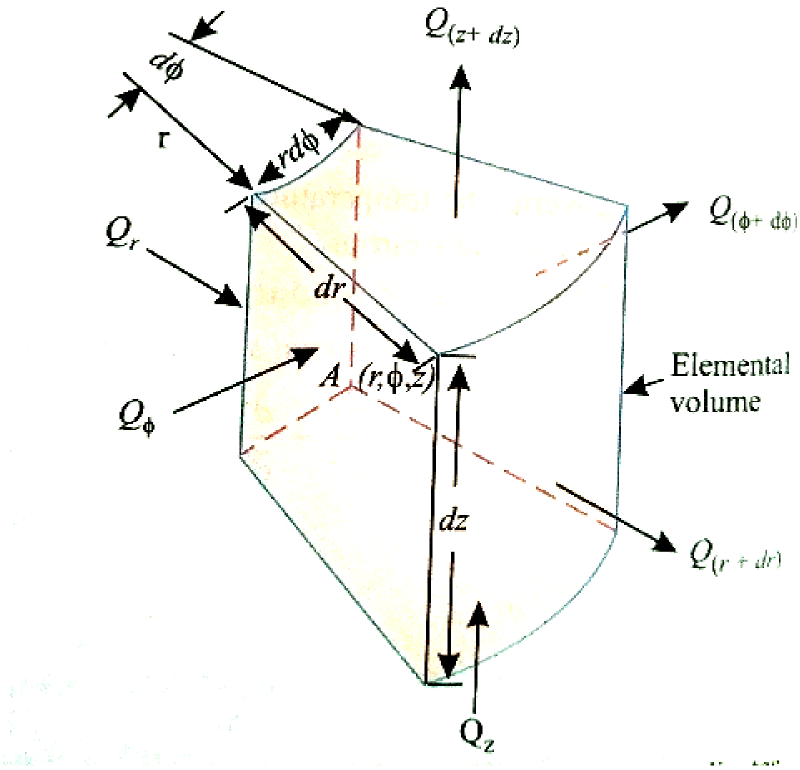
$$Q = \frac{[t_1 - t_{(n+1)}]}{\sum_1^n \frac{L}{KA}} \quad \text{...(vi)}$$

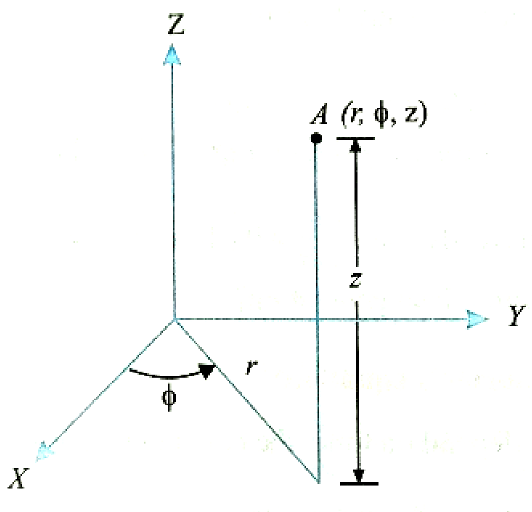
THERMAL RESISTANCE IN COMPSITE WALL-LINK BELOW

https://www.youtube.com/watch?v=pZMcUmf2_Ig

COMPOSITE WALL APPARATUS
DEPARTMENT OF MECHANICAL ENGINEERING,ABIT

GENERAL HEAT CONDUCTION EQUATION IN CYLINDRICAL COORDINATES





While dealing with problems of conduction of heat through systems having cylindrical geometries (ex.-rods and pipes) it is convenient to use cylindrical coordinates.

Consider an elemental volume having the coordinates (r, ϕ, z), for three dimensional heat conduction analysis, as shown in figure below.

The volume of the element=$rd\phi drdz$

Let, q_g =heat generation per unit volume per unit time.
Further, let us assume that k(thermal conductivity), ρ (density), c (specific heat) do not alter with position.

A. Net heat accumulated in the element due to conduction of heat from all the coordinate directions considered:

Heat flow in radial direction(x-ϕ) plane:

Heat influx, $Q'_r = -k(rd\phi dz)\frac{\partial t}{\partial r}.d\tau$ ……………………………………………(i)

Heat efflux, $Q'_{(r+dr)} = Q'_r + \frac{\partial}{\partial r}(Q_r)dr$ ……………………………………………(ii)

Hence, heat accumulation in the element due to heat flow in radial direction,

$$dQ'_r = Q_r' - Q'_{(r+dr)} \quad \text{[subtracting (ii) from (i)]}$$

$$= -\frac{\partial}{\partial r}(Q'_r)dr$$

$$= -\frac{\partial}{\partial r}[-k(rd\phi dz)\frac{\partial t}{\partial r}.d\tau]dr$$

$$= k(drd\phi dz)\frac{\partial}{\partial r}(r.\frac{\partial t}{\partial r})d\tau$$

$$= k(drd\phi dz)(r\frac{\partial^2 t}{\partial r^2} + \frac{\partial t}{\partial r})d\tau$$

$$= k(dr.rd\phi.dz)(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r})d\tau$$

Heat flow in tangential direction (r-z) plane:

Heat influx, $Q'_\phi = -k(drdz)\frac{\partial t}{r\partial \phi}d\tau$ ……………………………………………………(iii)

Heat efflux, $Q'_{(\phi+d\phi)} = Q'_\phi + \frac{\partial}{r\partial \phi}(Q'_\phi)rd\phi$ ………………………………………………(iv)

Heat accumulated in the element due to heat flow in tangential direction,

$$dQ'_\phi = Q'_\phi - Q'_{(\phi+d\phi)} \quad \text{……………………[subtracting (iv) from(iii)]}$$

$$= -\frac{\partial}{r\partial \phi}(Q_\phi)rd\phi$$

$$= -\frac{\partial}{r\partial \phi}[-k(drdz)\frac{\partial t}{r\partial \phi}.d\tau]rd\phi$$

$$= k(dr.d\phi.dz)\frac{\partial}{\partial \phi}(\frac{1}{r}.\frac{\partial t}{\partial \phi})d\tau$$

$= k(dr.rd\phi.dz)\frac{1}{r^2}.\frac{\partial^2 t}{\partial \phi^2}.d\tau$

Heat flow in axial direction(r-ϕ plane):

Heat influx, $Q'_z = -k(rd\phi dr)\frac{\partial t}{\partial z}d\tau$ ……………………………………………………………………………….(v)

Heat efflux, $Q'_{(z+dz)} =$, $Q'_z + \frac{\partial}{\partial z}(Q'_z)dz$ ……………………………………………………………………….(vi)

Heat accumulated in the element due to heat flow in axial direction,

$dQ'_z = Q'_z - Q'_{(z+dz)}$

$= -\frac{\partial}{\partial z}[-k(rd\phi dr)\frac{\partial t}{\partial z}.d\tau]dz$

$= k(dr.rd\phi.dz)\frac{\partial^2 t}{\partial z^2}.d\tau$

Net heat accumulated in the element.

$= k.dr.rd\phi.dz[\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}.\frac{\partial t}{\partial r} + \frac{1}{r^2}.\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}]d\tau$

B. Heat generated within the element(Q'_g):

The total heat generated within the element is given by

$Q'_g = q_g(dr.rd\phi.dz).d\tau$

C. Energy stored in the element:

The increase in thermal energy in the element is equal to

$= \rho(dr.rd\phi.dz).c.\frac{\partial t}{\partial \tau}.d\tau$

Now, (A)+(B)=(C) ……………………………Energy balance equation

Hence, $k.dr.rd\phi.dz[\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}.\frac{\partial t}{\partial r} + \frac{1}{r^2}.\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}]d\tau + q_g(dr.rd\phi.dz).d\tau = \rho(dr.rd\phi.dz).c.\frac{\partial t}{\partial \tau}.d\tau$

Dividing both sides by dr.rdϕ.dz.dτ, we have

$$k[\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}.\frac{\partial t}{\partial r} + \frac{1}{r^2}.\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}] + q_g = \rho c.\frac{\partial t}{\partial \tau}.$$

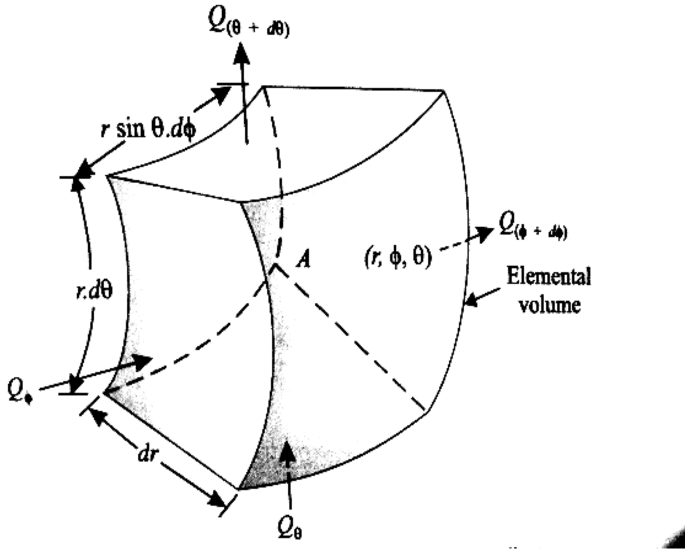
or, $$[\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}.\frac{\partial t}{\partial r} + \frac{1}{r^2}.\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}] + \frac{q_g}{k} = \frac{\rho c}{k}\frac{\partial t}{\partial \tau} = \frac{1}{\alpha}\frac{\partial t}{\partial \tau}$$

The above equation is the general heat conduction equation in cylindrical coordinates.

GENERAL HEAT CONDUCTION EQUATION IN CYLINDRICAL COORDINATES-LINK BELOW

https://www.youtube.com/watch?v=eGnq5dx2y_I

GENERAL HEAT CONDUCTION EQUATION IN SPHERICAL COORDINATES



Consider an elemental volume having the coordinates (r, ϕ, θ), for three dimensional heat conduction analysis as shown in figure above.

The volume of the element =dr.rdθ.rsinθdϕ

Let, q_g =heat generation per unit volume per unit time.

Further, let us assume that k(thermal conductivity), ρ (density), c (specific heat) do not alter with position

A. Net heat accumulated in the element due to conduction of heat from all the coordinate directions considered:

Heat flow through r-θ plane;ϕ-direction:

Heat influx, Q'_ϕ=-k(dr.rdθ)$\dfrac{\partial t}{r\sin\theta.\partial\phi}d\tau$ ……………………………………………(i)

Heat efflux, $Q'_{(\phi+d\phi)}$= Q'_ϕ+$\dfrac{\partial}{r\sin\theta.\partial\phi}(Q'_\phi)r\sin\theta d\phi$ ………………………..(ii)

Hence, heat accumulated in the element due to heat flow in the ϕ-direction,

$dQ'_\phi = Q'_\phi - Q'_{(\phi+d\phi)}$

$$= -\frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial\phi}(Q'_\phi)r\sin\theta.d\phi$$

=- $\frac{1}{r\sin\theta}.\frac{\partial}{\partial\phi}[-k(dr.rd\theta)\frac{1}{r\sin\theta}.\frac{\partial t}{\partial\phi}.d\tau]r\sin\theta.d\phi$

=k(dr.rdθ.rsinθ.dϕ) $\frac{1}{r^2\sin^2\theta}.\frac{\partial^2 t}{\partial\phi^2}d\tau$

Heat flow in r-ϕ plane,θ-direction:

Heat influx, Q'_θ=-k(dr.rsinθ.dϕ) $\frac{\partial t}{r\partial\theta}.d\tau$ ……………………………………………………………………(iii)

Heat efflux, $Q'_{(\theta+d\theta)}$= Q'_θ+ $\frac{\partial}{r\partial\theta}(Q'_\theta)rd\theta$ ……………………………………………………………………..(iv)

Hence, heat accumulated in the element due to heat flow in the θ-direction,

dq'_θ= Q'_θ- $Q'_{(\theta+d\theta)}$

=- $\frac{\partial}{r.\partial\theta}(Q'_\theta)r.d\theta$

=- $\frac{\partial}{r.\partial\theta}[-k(dr.r\sin\theta d\phi)\frac{\partial t}{r\partial\theta}.d\tau]r.d\theta$

= $\frac{k}{r}\frac{dr.rd\phi.rd\theta}{r}\frac{\partial}{\partial\theta}[\sin\theta.\frac{\partial t}{\partial\theta}].d\tau$

=k(dr.rdθ.rsinθ.dϕ) $\frac{1}{r^2\sin\theta}.\frac{\partial}{\partial\theta}[\sin\theta.\frac{\partial t}{\partial\theta}]d\tau$

Heat flow in θ-ϕ plane, r-direction:

Heat influx, Q'_r =-k(rdθ.rsinθ.dϕ) $\frac{\partial t}{\partial r}'\partial\tau$ ……………………………………………………………….(v)

Heat efflux, $Q'_{(r+dr)}$= Q'_r+ $\frac{\partial}{\partial r}(Q'_r)dr$ ……………………………………………………………………..(vi)

Hence, heat accumulation in the element due to heat flow in the r-direction,

dQ'_r=Q'_r-$Q'_{(r+dr)}$

=- $\frac{\partial}{\partial r}(Q'_r)dr$

=- $\frac{\partial}{\partial r}[-k(rd\theta.r\sin\theta.d\phi)\frac{\partial t}{\partial r}.d\tau]dr$

=kdθ.sinθ.dϕdr$\frac{\partial}{\partial r}[r^2.\frac{\partial t}{\partial r}]d\tau$

=k(dr.rdθ.rsinθ.dϕ)$\frac{1}{r^2}.\frac{\partial}{\partial r}[r^2.\frac{\partial t}{\partial r}]d\tau$

Net heat accumulated in the element

= kdr.rdθ.rsinθ.dϕ$[\frac{1}{r^2 \sin^2\theta}.\frac{\partial^2 t}{\partial \phi^2}+\frac{1}{r^2 \sin\theta}.\frac{\partial}{\partial \theta}(\sin\theta.\frac{\partial t}{\partial \theta})+\frac{1}{r^2}.\frac{\partial}{\partial r}(r^2.\frac{\partial t}{\partial r})]d\tau$

B. Heat generated within the element(Q'_g):

The total heat generated within the element is given by.

Q'_g=q_g(dr.rdθ.rsinθ.dϕ)dτ

C. Energy stored in the element:

The increase in thermal energy in the element is equal to,

ρ(dr.rdθ.rsinθ.dϕ)c.$\frac{\partial t}{\partial \tau}.d\tau$

Now (A)+(B)=(C)

Hence, kdr.rdθ.rsinθ.dϕ$[\frac{1}{r^2 \sin^2\theta}.\frac{\partial^2 t}{\partial \phi^2}+\frac{1}{r^2 \sin\theta}.\frac{\partial}{\partial \theta}(\sin\theta.\frac{\partial t}{\partial \theta})+\frac{1}{r^2}.\frac{\partial}{\partial r}(r^2.\frac{\partial t}{\partial r})]d\tau$

+q_g(dr.rdθ.rsinθ.dϕ)dτ= ρ(dr.rdθ.rsinθ.dϕ)c.$\frac{\partial t}{\partial \tau}.d\tau$

Dividing both sides by k(dr.rdθ.rsinθ.dϕ)dτ,we get

$$[\frac{1}{r^2 \sin^2\theta}.\frac{\partial^2 t}{\partial \phi^2}+\frac{1}{r^2 \sin\theta}.\frac{\partial}{\partial \theta}(\sin\theta.\frac{\partial t}{\partial \theta})+\frac{1}{r^2}.\frac{\partial}{\partial r}(r^2.\frac{\partial t}{\partial r})]+\frac{q_g}{k}=\frac{\rho c}{k}.\frac{\partial t}{\partial \tau}=\frac{1}{\alpha}.\frac{\partial t}{\partial \tau}$$

This is the general heat conduction equation in spherical coordinates.

GENERAL HEAT CONDUCTION EQUATION IN SPHERICAL COORDINATES-LINK BELOW

https://www.youtube.com/watch?v=sIXku2DqUzQ

HEAT CONDUCTION THROUGH A PLANE WALL-VARIABLE THERMAL CONDUCTIVITY

Let the thermal conductivity vary with temperature according to the relation

$K=k_0 (1+\beta t)$

[In most cases ,the thermal conductivity is found to vary linearly with temperature]

Where, k_0 = Thermal conductivity at zero temperature

When the effect of temperature on thermal conductivity is considered , the Fourier's equation,

$Q=-kA\frac{dt}{dx}$ is written as

$Q=-k_0 (1+\beta t)\frac{dt}{dx}.A$

Or, $$\frac{Q}{A}.dx = -k_0(1+\beta t)dt$$

Or, $$\frac{Q}{A}\int_0^L dx = -k_0\int_{t_1}^{t_2}(1+\beta t)dt$$

Or, $$\frac{QL}{A} = -k_0[t+\frac{\beta}{2}t^2]_{t_1}^{t_2}$$

Or, $$\frac{QL}{A} = -k_0[(t_2-t_1)+\frac{\beta}{2}(t_2{}^2-t_1{}^2)]$$

$$= k_0[(t_1-t_2)+\frac{\beta}{2}(t_1-t_2)(t_1+t_2)]$$

$$= k_0[1+\frac{\beta}{2}(t_1+t_2)](t_1-t_2)$$

$$= k_0[1+\beta t_m].(t_1-t_2)$$

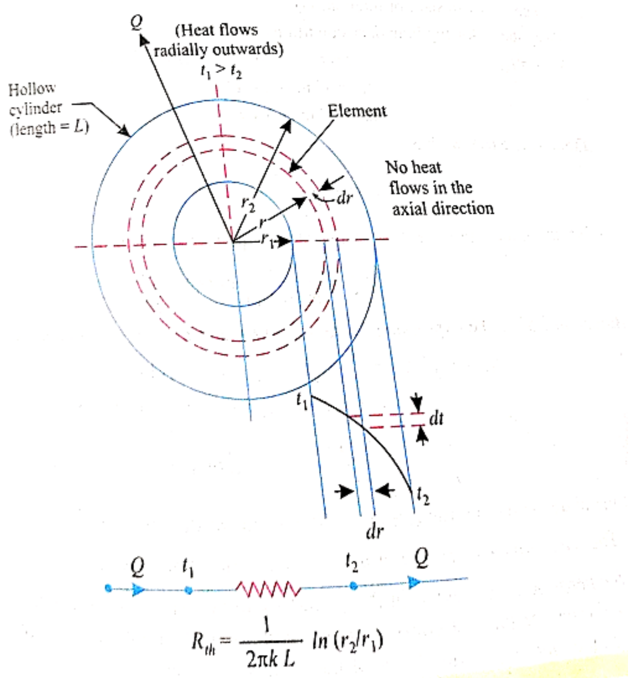
Hence, $$Q = k_0[1+\beta t_m].\frac{A(t_1-t_2)}{L}$$

But, $k_m = k_0 (1+\beta t_m)$

Hence, $$Q=k_mA[\frac{t_1-t_2}{l}]$$

Where k_m is known as mean thermal conductivity of the wall material.

HEAT CONDUCTION THROUGH A HOLLOW CYLINDER



Consider a hollow cylinder made of material having constant thermal conductivity and insulated at both ends.

Let, r_1,r_2 = Inner and outer radii:

t_1,t_2 =temperatures of inner and outer surfaces, and

k=constant thermal conductivity.

Consider an element at radius 'r' and thickness 'dr' for a length of the hollow cylinder through which heat is transmitted. Let dt be the temperature drop over the element.

Area through which heat is transmitted , A=2 $\pi.r.L$

Path length=dr (over which the temperature falls dt)

Hence, $Q=-kA.(\frac{dt}{dr})$

$=-k.\ 2\pi.r.L\frac{dt}{dr}$ per unit time

Or, $Q.\frac{dr}{r} = -k.2\pi L dt$

Integrating both sides, we ge

$$Q\int_{r_1}^{r_2}\frac{dr}{r} = -k2\pi L\int_{t_1}^{t_2} dt$$

Or, $$Q[\ln(r)]_{r_1}^{r_2} = -k2\pi L[t]_{t_1}^{t_2}$$

Or, $$Q.\ln(\frac{r_2}{r_1}) = -k.2\pi L(t_2 - t_1) = k2\pi L(t_1 - t_2)$$

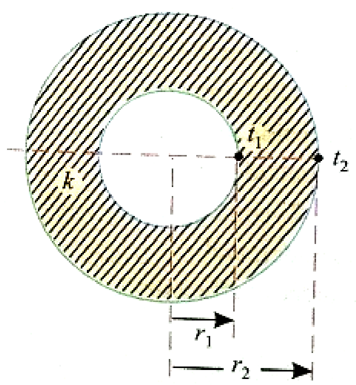
Hence, $$Q = \frac{k.2\pi L(t_1 - t_2)}{\ln(\frac{r_2}{r_1})} = \frac{(t_1 - t_2)}{\left[\frac{\ln({r_2}/{r_1})}{2\pi kL}\right]}$$

HEAT CONDUCTION THROUGH HOLLOW CYLINDER AND SPHERE-LINK BELOW

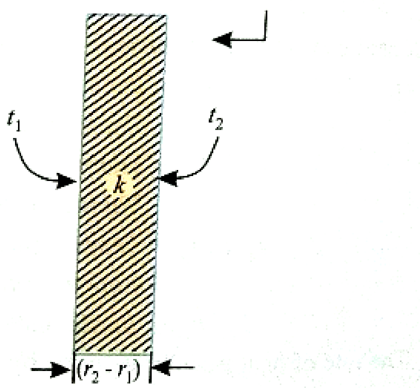
https://www.youtube.com/watch?v=PkcGgaxujJg

https://www.youtube.com/watch?v=hX97HMRW5Gg

LOGARITHMIC MEAN AREA FOR THEHOLLOW CYLINDER



(a) Hollow cylinder



(b) Plane wall

Let us consider a hollow cylinder having r_1 and r_2 be the inner and outer radius and k be the thermal conductivity an d let, A be the area.

Let (r_2-r_1) be the thickness of a wall made of same material and let, A_m be the area of the wall such that heat flow Q is same in both cylinder and the wall.

Hence, $$Q=\frac{(t_1-t_2)}{\frac{\ln(r_2/r_1)}{2\pi kL}}$$ ………………………………..heat flow through cylinder.

$$Q=\frac{(t_1-t_2)}{\frac{(r_2-r_1)}{kA_m}}$$ ……………………………………..heat flow through plane wall.

Hence, $$\frac{(t_1-t_2)}{\frac{\ln(r_2/r_1)}{2\pi kL}}=\frac{(t_1-t_2)}{\frac{(r_2-r_1)}{kA_m}}$$

Or, $$\frac{\ln(r_2/r_1)}{2\pi kL}=\frac{(r_2-r_1)}{kA_m}$$

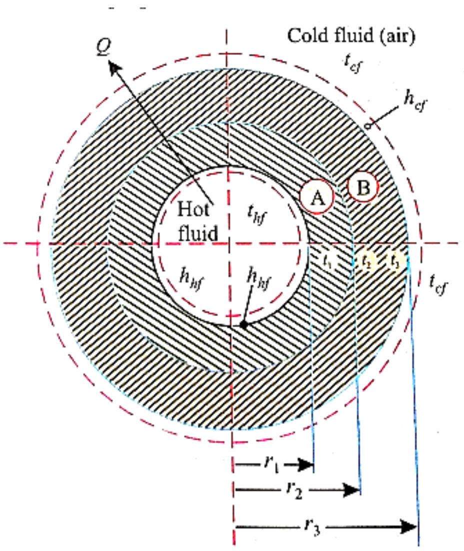
Or, $$A_m=\frac{2\pi L(r_2-r_1)}{\ln(r_2/r_1)}=\frac{2\pi Lr_2-2\pi Lr_1}{\ln(2\pi Lr_2/2\pi Lr_1)}$$

Or, $$A_m=\frac{A_0-A_i}{\ln(A_0-A_i)}$$

Where A_i and A_0 are inside and outside surface areas of the cylinder.

The expression is known as logarithmic mean area of the plane wall and the hollow cylinder. By the use of this expression a cylinder can be transformed into a plane wall and the problem can be solved easily.

HEAT CONDUCTION THROUGH A COMPOSITE CYLINDER



Consider flow of heat through a composite cylinder as shown in the figure.

Let, t_{hf} = The temperature of hot fluid flow inside a cylinder,

t_{cf} =The temperature of cold fluid(atmospheric air),

k_A=Thermal conductivity of the inside layer A,

k_B=Thermal conductivity of the outside layer B,

t_1,t_2,t_3=Temperatures at the points 1,2 and3

L=Length of the composite cylinder, and

h_{hf},h_{cf}=Inside and outside heat transfer coefficients.

The rate of heat transfer is given by

$$Q=h_{hf}.2\pi r_1.L(t_{hf}-t_1)=\frac{k_A.2\pi L(t_1-t_2)}{\ln(r_2/r_1)}=\frac{k_B.2\pi L(t_2-t_3)}{\ln(r_3/r_2)}=h_{cf}.2\pi r_3.L(t_3-t_{cf})$$

Rearranging the above expression, we get

$$t_{hf}-t_1=\frac{Q}{h_{hf}.r_1.2\pi L} \quad \text{...(i)}$$

$$t_1 - t_2 = \frac{Q}{\frac{k_A.2\pi L}{\ln(r_2/r_1)}} \quad \text{...(ii)}$$

$$t_2 - t_3 = \frac{Q}{\frac{k_B.2\pi L}{\ln(r_3/r_2)}} \quad \text{...(iii)}$$

$$t_3 - t_{cf} = \frac{Q}{h_{cf}.r_3.2\pi L} \quad \text{...(iv)}$$

Adding (i), (ii), (iii) and (iv), we have

$$\frac{Q}{2\pi L}\left[\frac{1}{h_{hf}.r_1} + \frac{1}{\frac{k_A}{\ln(r_2/r_1)}} + \frac{1}{\frac{k_B}{\ln(r_3/r_2)}} + \frac{1}{h_{cf}.r_3}\right] = t_{hf} - t_{cf}$$

Hence, Q= $\dfrac{2\pi L(t_{hf} - t_{cf})}{\left[\frac{1}{h_{hf}.r_1} + \frac{1}{\frac{k_A}{\ln(r_2/r_1)}} + \frac{1}{\frac{k_B}{\ln(r_3/r_2)}} + \frac{1}{h_{cf}.r_3}\right]}$

$$Q = \frac{2\pi L(t_{hf} - t_{cf})}{\left[\frac{1}{h_{hf}r_1} + \frac{\ln(r_2/r_1)}{k_A} + \frac{\ln(r_3/r_2)}{k_B} + \frac{1}{h_{cf}.r_3}\right]}$$

If there are 'n' concentric cylinders, then

$$Q = \frac{2\pi L(t_{hf} - t_{cf})}{\left[\frac{1}{h_{hf}r_1} + \sum_{n=1}^{n=n}\frac{1}{k_n}\ln(r_{n+1}/r_n) + \frac{1}{h_{cf}.r_{n+1}}\right]}$$

If inside and outside heat transfer coefficient are not considered then the above equation can be written as

$$Q = \frac{2\pi L(t_1 - t_{n+1})}{\sum_{n=1}^{n=n}\frac{1}{k_n}\ln(r_{n+1}/r_n)}$$

HEAT CONDUCTION THROUGH HOLLOW SPHERE

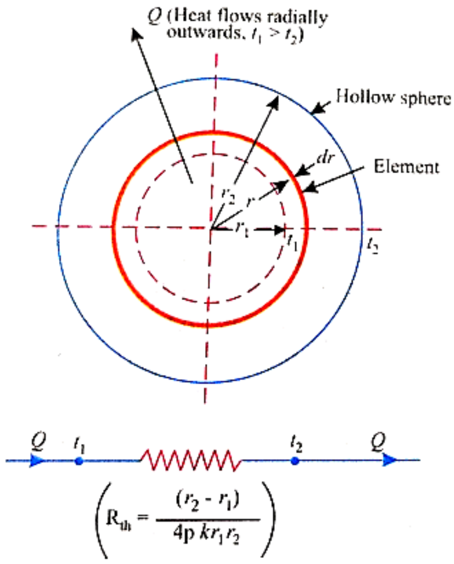


Fig. Steady state conduction through a hollow sphere.

Consider a hollow sphere made of material having constant thermal conductivity.

Let, r_1, r_2 =Inner and outer radii,

t_1,t_2 =Temperatures of inner and outer surfaces, and

k= Constant thermal conductivity of the material.

Consider a small element of thickness dr at any radius r.

Area through which the heat is transmitted, A=4 πr^2

Hence, $Q = -k.4\pi r^2 . \frac{dt}{dr}$

Rearranging and integrating the above equation, we obtain

$$Q\int_{r_1}^{r_2} \frac{dr}{r^2} = -4\pi k \int_{t_1}^{t_2} dt$$

Or, $$Q\left[\frac{r^{-2+1}}{-2+1}\right]_{r_1}^{r_2} = -4\pi k [t]_{t_1}^{t_2}$$

Or, $$-Q\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = -4\pi k (t_2 - t_1)$$

Or, $$\frac{Q(r_2-r_1)}{r_1r_2}=4\pi k(t_1-t_2)$$

Or, $$Q=\frac{4\pi kr_1r_2(t_1-t_2)}{(r_2-r_1)}=\frac{(t_1-t_2)}{\left[\frac{(r_2-r_1}{4\pi kr_1r_2}\right]}$$

LOGARITHMIC MEAN AREA FOR THE HOLLOW SPHERE

We know, $$Q_{sphere}=\frac{(t_1-t_2)}{\left[\frac{(r_2-r_1}{4\pi kr_1r_2}\right]}$$

$$Q_{planewall}=\frac{(t_1-t_2)}{\left[\frac{(r_2-r_1}{kA_m}\right]}$$

A_m is so chosen that the heat flow through sphere and plane wall will be equal for the same thermal potential.

Hence, $Q_{sphere}=Q_{plane\ wall}$

$$\frac{(t_1-t_2)}{\left[\frac{(r_2-r_1}{4\pi kr_1r_2}\right]}=\frac{(t_1-t_2)}{\left[\frac{(r_2-r_1}{kA_m}\right]}$$

$$\frac{r_2-r_1}{4\pi kr_1r_2}=\frac{r_2-r_1}{kA_m}$$

Or, $A_m=4\pi r_1r_2$

Or, $$A_m^2=(4\pi r_1r_2)^2=(4\pi r_1^2)x(4\pi r_2^2)$$

Or, $$A_m^2=A_ixA_0$$

Or, $$A_m=\sqrt{A_iA_0}$$

Or, $$A_m=4\pi r_m^2=4\pi r_1r_2$$

HEAT CONDUCTION THROUGH A COMPOSITE SPHERE

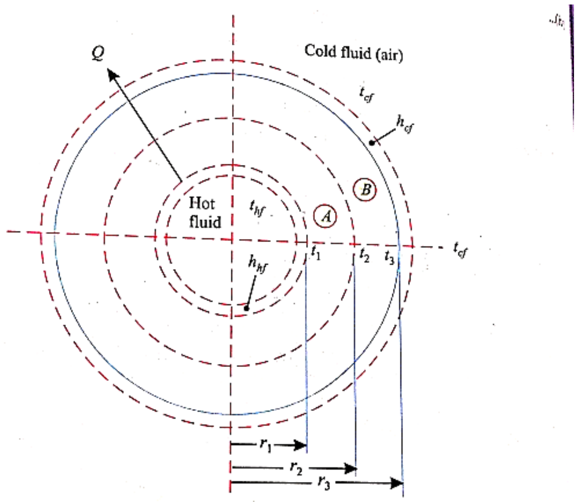


Fig. Steady state conduction through a composite sphere.

Consider the above figure as cross-section of a composite sphere, the heat flow equation is given as

$$Q = h_{hf}.4\pi r_1^2(t_{hf} - t_1) = \frac{4\pi k_A r_1 r_2(t_1 - t_2)}{(r_2 - r_1)} = \frac{4\pi k_B r_2 r_3(t_2 - t_3)}{(r_3 - r_2)} = h_{cf}.4\pi r_3^2(t_3 - t_{cf})$$

By rearranging the above equation we have

$$t_{hf} - t_1 = \frac{Q}{h_{hf}.4\pi r_1^2} \quad \text{...(i)}$$

$$t_1 - t_2 = \frac{Q(r_2 - r_1)}{4\pi k_A.r_1 r_2} \quad \text{...(ii)}$$

$$t_2 - t_3 = \frac{Q(r_3 - r_2)}{4\pi k_B.r_2 r_3} \quad \text{..(iii)}$$

$$t_3 - t_{cf} = \frac{Q}{h_{cf}.4\pi r_3^2} \quad \text{..(iv)}$$

Adding (i), (ii), (iii),(iv), we get

$$\frac{Q}{4\pi}\left[\frac{1}{h_{hf}.r_1^2} + \frac{(r_2 - r_1)}{k_A.r_1 r_2} + \frac{(r_3 - r_2)}{k_B.r_2 r_3} + \frac{1}{h_{cf}.r_3^2}\right] = t_{hf} - t_{cf}$$

Hence, Q= $\dfrac{4\pi(t_{hf}-t_{cf})}{\left[\dfrac{1}{h_{hf}.r_1^2}+\dfrac{(r_2-r_1)}{k_A.r_1r_2}+\dfrac{(r_3-r_2)}{k_B.r_2r_3}+\dfrac{1}{h_{cf}.r_3^2}\right]}$

If there are n concentric spheres then the above equation can be written a follows,

$$Q=\frac{4\pi(t_{hf}-t_{cf})}{\left[\dfrac{1}{h_{hf}r_1^2}+\sum_{n=1}^{n=n}\left(\dfrac{r_{n+1}-r_n}{k_n.r_n.r_{n+1}}\right)+\dfrac{1}{h_{cf}.r_{n+1}^2}\right]}$$

OVERALL HEAT TRANSFER COEFFICIENT (COMBINED MODE OF HEAT TRANSFER)

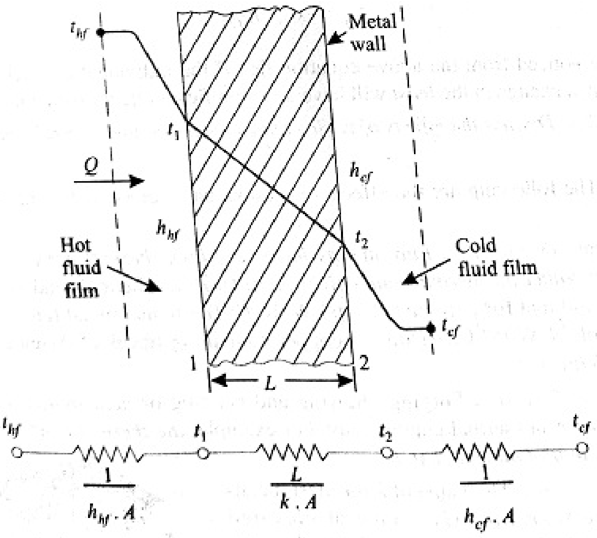


Fig. : The overall heat transfer through a plane wall.

When there is a problem of fluid to fluid heat transfer across a metal boundary, it is usual to adopt an overall heat transfer coefficient U which gives the heat transmitted per unit area per unit time per degree temperature difference between the bulk fluids on each side of the metal.

Referring to the figure above,

Let, L=Thickness of the metal wall,

K= Thermal conductivity of the wall material,

t_1= Temperature of surface-1,

t_2 =Temperature of surface-2,

t_{hf}=Temperature of hot fluid,

t_{cf}= Temperature of the cold fluid,

h_{hf} =Heat transfer coefficient from hot fluid to metal surface, and

h_{cf}= Heat transfer coefficient from metal surface to cold fluid.

The equations of heat flow through the fluid and the metal surface is given by

Q= $h_{hf} A(t_{hf} - t_1)$ ……………………………………………….(i)

Q= $\frac{kA(t_1 - t_2)}{L}$ ……………………………………………………(ii)

Q= $h_{cf} A(t_2 - t_{cf})$ ………………………………………………….(iii)

By rearranginf (i),(ii), and (iii), we get,

$$t_{hf} - t_1 = \frac{Q}{h_{hf}.A} \quad \text{………………………………………………………(iv)}$$

$$t_1 - t_2 = \frac{QL}{kA} \quad \text{…………………………………………………………….(v)}$$

$$t_2 - t_{cf} = \frac{Q}{h_{cf} A} \quad \text{……………………………………………………….(vi)}$$

Adding (iv),(v) and(vi) we get,

$$t_{hf} - t_{cf} = Q\left[\frac{1}{h_{hf}.A} + \frac{L}{kA} + \frac{1}{h_{cf}.A}\right]$$

OR, Q= $\dfrac{A(t_{hf} - t_{cf})}{\frac{1}{h_{hf}} + \frac{L}{k} + \frac{1}{h_{cf}}}$

If U is theoverall coefficient of heat transfer, then

Q=U.A(t_{hf}-t_{cf})= $\dfrac{A(t_{hf} - t_{cf})}{\frac{1}{h_{hf}} + \frac{L}{k} + \frac{1}{h_{cf}}}$

Or, U= $\dfrac{1}{\frac{1}{h_{hf}} + \frac{L}{k} + \frac{1}{h_{cf}}}$

COMBINED MODE OF HEAT TRANSFER(OVERALL HEAT TRANSFER COEFFICIENT) IN A PIPE-LINK BELOW

https://www.youtube.com/watch?v=D78wOJvzCsY

CRITICAL THICKNESS OF INSULATION

INSULATION-DEFINITION.

A material which retards the flow of heat with reasonable effectiveness is known as 'insulation'. Insulation serves the following two purposes:

(i) It prevents the heat flow from the system to the surroundings;

(ii) It prevents the heat flow from the surroundings to the system.

APPLICATIONS

(i) Boilers and steam pipes,

(ii) Air conditioning systems,

(iii) Food preserving stores and refrigerators,

(iv) Insulating bricks,

(v) Preservation of liquid gases etc.

CRITICAL THICKNESS OF INSULATION

- The addition of insulation always increases the conductive thermal resistance. But when the total thermal resistance is made of conductive thermal resistance[$(R_{th})_{cond.}$] and convective thermal resistance [$(R_{th})_{conv.}$], the addition of insulation in some cases may reduce the convective thermal resistance due to increase in surface area, as in the case of a cylinder and sphere ,and the total thermal resistance may actually decrease resulting in increased heat flow.
- It may be shown that the thermal resistance actually decreases and then increases in some cases.
- "The thickness up to which heat flow increases and after which heat flow decreases is termed as Critical thickness. In case of cylinders and spheres it is called 'Critical radius' .

Critical thickness of insulation for cylinder:

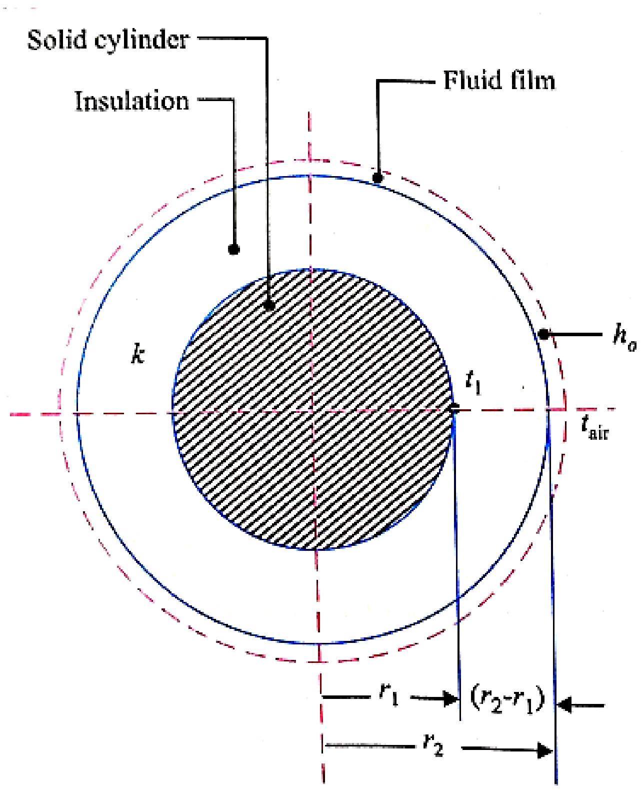


Fig. . Critical thickness of insulation for cylinder.

Consider a solid cylinder of radius r_1 insulated with an insulation of thickness (r_2-r_1) as shown in the figure above.

Let, L=Length of cylinder,

t_1=Surface temperature of cylinder,

t_{air}= Temperature of air,

h_o= Heat transfer coefficient at the outer surface of the insulation, and

k=Thermal conductivity of insulating material.

Then the rate of heat transfer from the surface of a solid cylinder to the surroundings is given by

$$Q = \frac{2\pi L(t_1 - t_{air})}{\frac{\ln(r_2/r_1)}{k} + \frac{1}{h_0 r_2}}$$

From above equation it is evident that as r_2 increases , the factor $\frac{\ln(r_2/r_1)}{k}$ increases but the factor $\frac{1}{h_o r_2}$ decreases. Thus Q becomes maximum when the denominator $\left[\frac{\ln(r_2/r_1)}{k}+\frac{1}{h_o r_2}\right]$ becomes minimum. The required condition is

$$\frac{d}{dr_2}\left[\frac{\ln(r_2/r_1)}{k}+\frac{1}{h_o r_2}\right]=0 \qquad \text{(}r_2\text{ being the only variable)}$$

Hence, $\frac{1}{k}.\frac{1}{r_2}+\frac{1}{h_o}\left(-\frac{1}{r_2^2}\right)=0$

Or, $\frac{1}{k}-\frac{1}{h_o r_2}=0$

Or, $h_o r_2 = k$

Or, $r_2(=r_c)=\frac{k}{h_o}$

CRITICAL THICKNESS OF INSULATION-LINK BELOW

https://www.youtube.com/watch?v=5P5S_MzdcS4

Critical thickness of insulation for sphere:

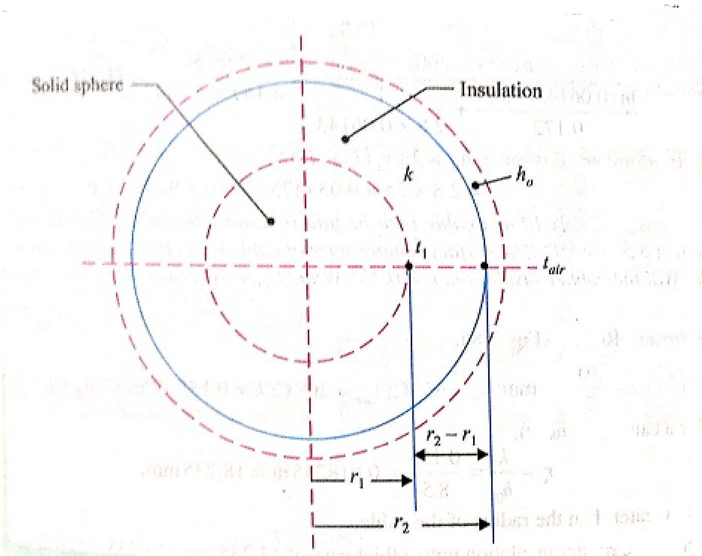


Fig. Critical thickness of insulation for sphere.

Referring to figure, the equation of heat flow through a sphere with insulation is given as

$$Q = \frac{(t_1 - t_{air})}{\left[\frac{r_2 - r_1}{4\pi k r_1 r_2}\right] + \frac{1}{4\pi r_2^2 . h_o}}$$

Adding the same procedure as that of a cylinder, we have

$$\frac{d}{dr_2}\left[\frac{r_2 - r_1}{4\pi k r_1 r_2} + \frac{1}{4\pi r_2^2 . h_o}\right] = 0$$

Or, $$\frac{d}{dr_2}\left[\frac{1}{kr_1} - \frac{1}{kr_2} + \frac{1}{r_2^2 . h_o}\right] = 0$$

Or, $$\frac{1}{kr_2^2} - \frac{2}{r_2^3 h_o} = 0$$

Or, $$r_2^3 h_o = 2kr_2^2$$

Or, $$r_2 (= r_c) = \frac{2k}{h_o}$$

HEAT CONDUCTION WITH INTERNAL HEAT GENERATION

Following are some of the cases where heat generation and heat conduction are encountered:

(i) Fuel rods-nuclear reactor,

(ii) Electrical conductors,

(iii) Chemical and combustion processes'

(iv) Drying and setting of concrete.

PLANE WALL WITH UNIFORM HEAT GENERATION

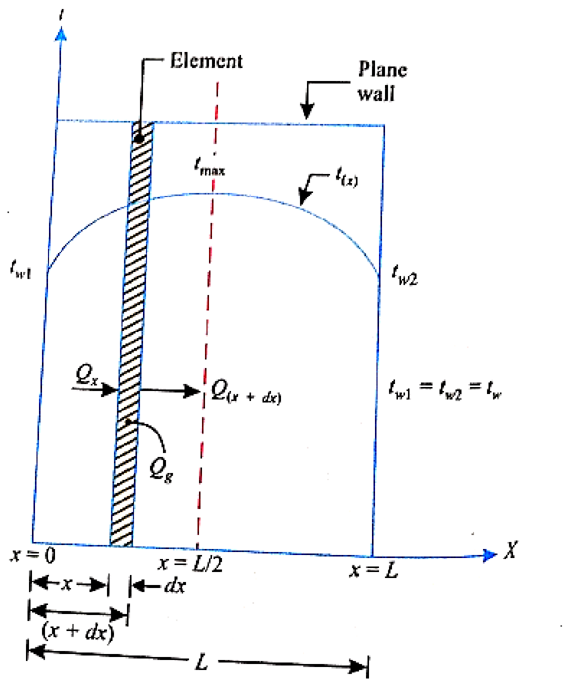


Fig. Plane wall uniform heat generation. Both the surfaces maintained at a common temperature.

Consider a plane wall of thickness L(small in comparison to other dimension) of uniform thermal conductivity k and in which heat sources are uniformly distributed in the whole volume.

Let the wall surfaces are maintained at temperatures t_1 and t_2.

Let us assume that heat flow is one dimensional, under steady state conditions and there is a uniform volumetric heat generation within the wall.

Consider a element of thickness at a distance x from the left hand face of the wall.

Heat conducted in at distance x,

$$Q_x = -kA\frac{dt}{dx}$$

Heat generated in the element,

$$Q_g = A.dx.q_g$$

(Where q_g = heat generated per unit volume per unit time in the element)

Heat conducted out at distance(x+dx),

$$Q_{(x+dx)} = Q_x + \frac{d}{dx}(Q_x)dx$$

Ass Q_g represents an energy increase in the volume element, an energy balance on the element of thick dx is given by

$$Q_x + Q_g = Q_{(x+dx)}$$

$$= Q_x + \frac{d}{dx}(Q_x)dx$$

Or, $$Q_g = \frac{d}{dx}(Q_x)dx$$

Or, $$q_g.A.dx = \frac{d}{dx}\left[-kA\frac{dt}{dx}\right]dx$$

$$= -kA.\frac{d^2t}{dx^2}.dx$$

Or, $$\frac{d^2t}{dx^2} + \frac{q_g}{k} = 0$$

The first and second integration of above equation gives,

$$\frac{dt}{dx} = -\frac{q_g}{k}x = C_1$$

$$t = -\frac{q_g}{2k}.x^2 + C_1x + C_2$$

CASE I. BOTH THE SURFACES HAVE THE SAME TEMPERATURE:

At x=0 $t=t_1=t_w$, and

At x=L $t=t_2=t_w$

(where t_w = temperature of the wall surface)

Using these boundary conditions, we get

$C_2=t_w$ and $C_1=\frac{q_g}{2k}L$

Substituting these values of C_1 and C_2 ,we have

$$t = -\frac{q_g}{2k}x^2 + \frac{q_g}{2k}.L.x + t_w$$

$$t = \frac{q_g}{2k}(L-x)x + t_w$$

In order to determine the location of the maximum temperature , differentiating the above equation with respect to x and equating the derivative to zero, we have

$$\frac{dt}{dx} = \frac{q_g}{2k}(L-2x) = 0$$

Since, $\frac{q_g}{2k} \neq 0, therefore,$

L-2x=0 or x=L/2

The maximum temperature occurs at x=L/2 and its value equals

$$t_{\max} = \left[\frac{q_g}{2k}(L-x)x\right]_{x=L/2} + t_w$$

Or, $= \left[\frac{q_g}{2k}(L-L/2)L/2\right] + t_w$

i.e. $t_{\max} = \frac{q_g}{8k}.L^2 + t_w$

Heat transfer then takes place towards both the surfaces, and for eah urface it is given by

$$Q = -kA\left(\frac{dt}{dx}\right)_{x=0 or x=L}$$

$$= -kA\left[\frac{q_g}{2k}(L-2x)\right]_{x=0 or x=L}$$

I.e. $Q = \frac{AL}{2}.q_g$

When both the surfaces are considered,

$$Q = 2X\frac{AL}{2}q_g = A.L.q_g$$

Also heat conducted to each wall surface is further dissipated to the surrounding atmosphere at temperature t_a.

Thus, $\frac{AL}{2}.q_g = hA(t_w - t_a)$

Or, $t_w = t_a + \frac{q_g}{2h}.L$

Hence, $t = t_a + \frac{q_g}{2h}.L + \frac{q_g}{2k}(L-x)x$

At x=L/2. i.e. at the midplane:

$$t = t_{\max} = t_a + \frac{q_g}{2h}.L + \frac{q_g L^2}{8k}$$

Or, $t_{\max} = t_a + q_g\left[\frac{L}{2h} + \frac{L^2}{8k}\right]$

INSULATED WALL

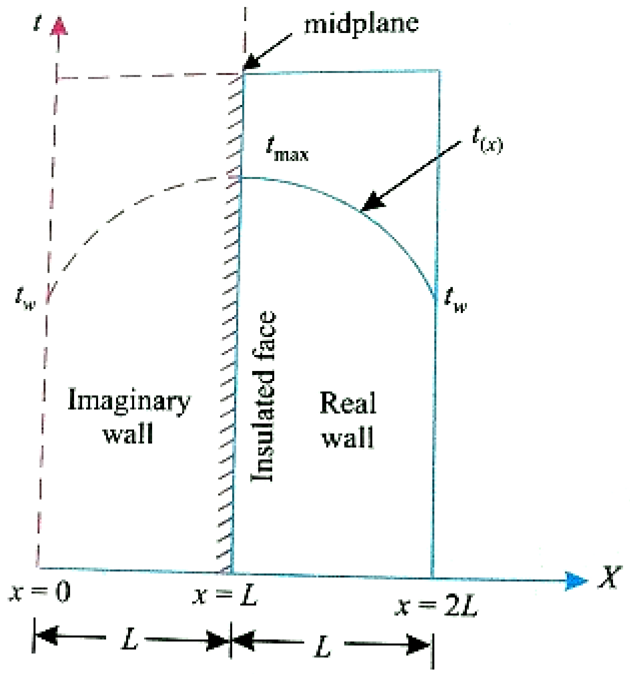


Fig. Heat conduction in an insulated wall.

The following boundary conditions apply in the full hypothetical wall of thickness 2L;

At x=L dt/dx=0

At x=2L $t=t_w$

The location x=L refers to the mid-plane of the hypothetical wall or insulated face of a given wall)Equations for temperature distribution and maximum temperature at the mid-plane(insulated end of the gien wall) respectively can bewritten as

$$t = \frac{q_g}{2k}(2L - x)x + t_w$$

$$t_{\max} = \frac{q_g}{2k}L^2 + t_w$$

[Substituting L=2L]

CASE II . BOTH THE SURFACESOF THE WALL HAVE DIFFERENT TEMPERATURES:

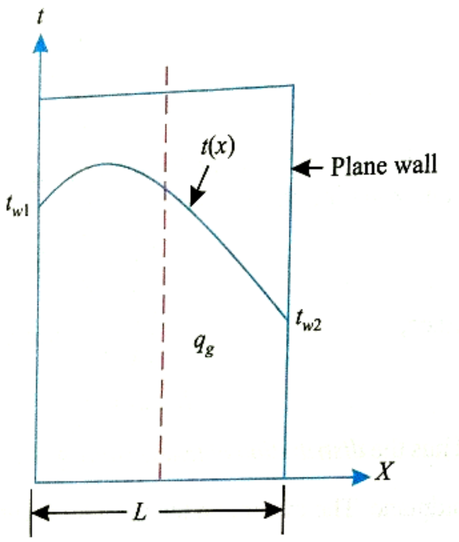


Fig. Plane wall with uniform heat generation—Both the surfaces of the wall having different temperatures.

Referring to the above figure , the boundary conditions are:

At x=0 $t = t_{w_1}$

At x=L $t = t_{w_2}$

Substituting these values in the equation

$t = -\frac{q_g}{2k}.x^2 + C_1 x + C_2$, we obtain the values of constant C_1 and C_2 as:

$C_2 = t_{w1}$ $C_1 = \frac{t_{w2} - t_{w1}}{L} + \frac{q_g}{2k}.L$

Inserting these values we get,

$$t = -\frac{q_g}{2k}x^2 + \frac{t_{w2} - t_{w1}}{L}x + \frac{q_g}{2k}L.x + t_{w1}$$

$$= \frac{q_g}{2k}Lx - \frac{q_g}{2k}x^2 + \frac{x}{L}(t_{w2} - t_{w1}) + t_1$$

$$t = \left[\frac{q_g}{2k}(L-x) + \frac{t_{w2} - t_{w1}}{L}\right]x + t_{w1}$$

HEAT TRANSFER FROM EXTENDED SURFACES (FINS)

Introduction

- Whenever the available surface is found inadequate to transfer the required quantity of heat with the available temperature drop and convective heat transfer coefficient, extended surfaces or fins are used.

- The finned surfaces are widely used in

(i) Economizers for steam power plants,

(ii) Radiators of automobiles,

(iii) Air-cooled engine cylinder heads ,

(iv) Cooling coils and condenser coils in refrigerators and air conditioners,

(v) Small capacity compressors,

(vi) Electric motor bodies,

(vii) Transformers and electronic equipments etc.

- Some common types of fin configurations are shown below

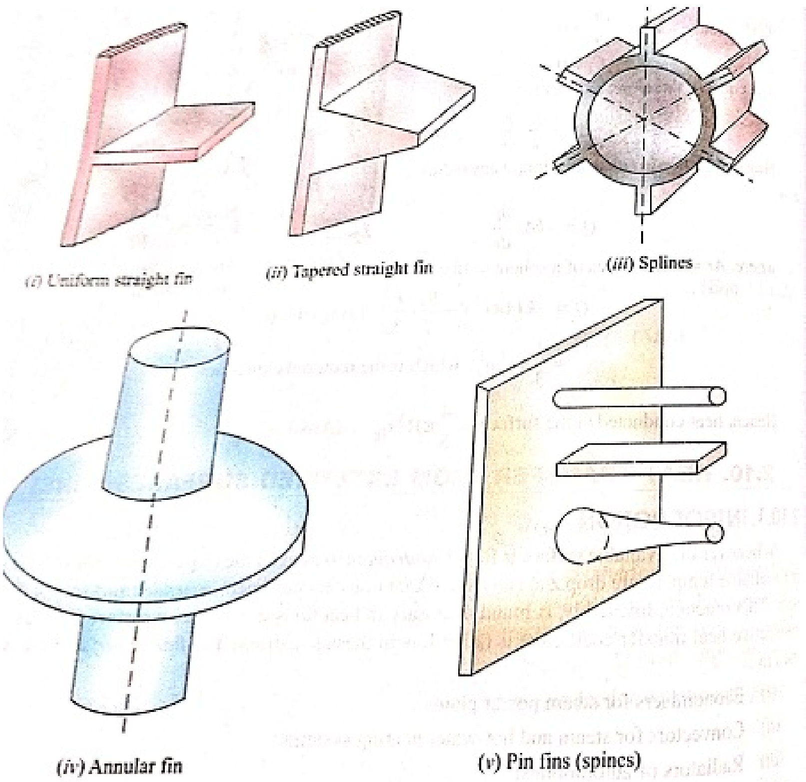
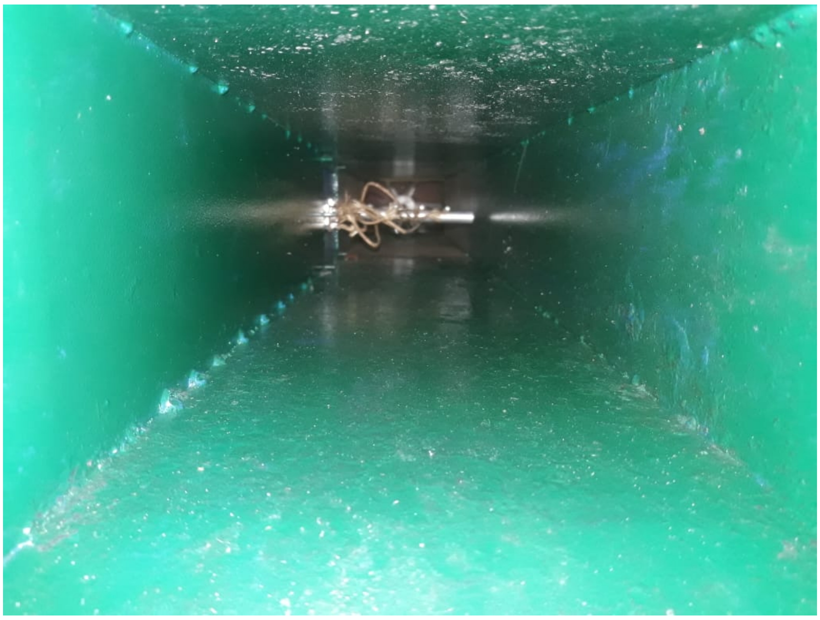


Fig. Common types of fin configurations.

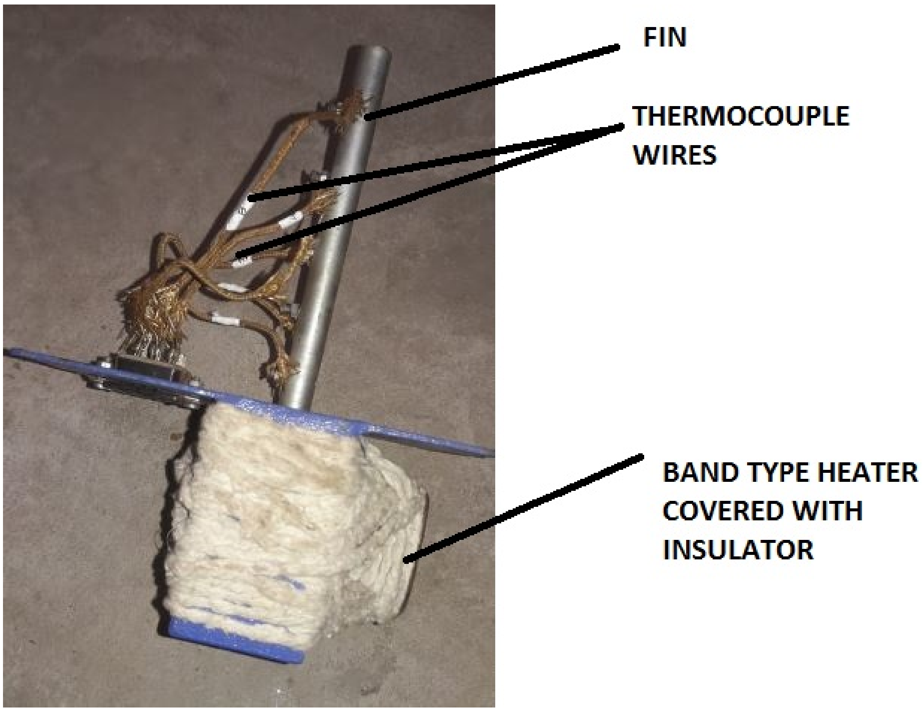
Pin Fin Heat sink is used to cool the processor

PIN FIN APPARATUS

DEPARTMENT OF MECHANICAL ENGINEEERING , ABIT.



PIN FIN



FIN AND HEATER

- For the proper design of fins , the knowledge of temperature distribution along the fin is necessary.

  INTRODUCTION TO FINS-LINK BELOW

  https://www.youtube.com/watch?v=c9HdoCZNlTg

- The following assumptions are made for the analysis of heat flow through the fin:

1. Steady state heat conduction.
2. No heat generation within the fin.
3. Uniform heat transfer coefficient (h) over the entire surface of fin.
4. Homogeneous and isotropic fin material (I,e. thermal conductivity of material is constant).
5. Negligible contact thermal resistance.
6. Heat conduction one-dimensional.
7. Negligible radiation.

HEAT FLOW THROUGH" RECTANGULAR FIN"

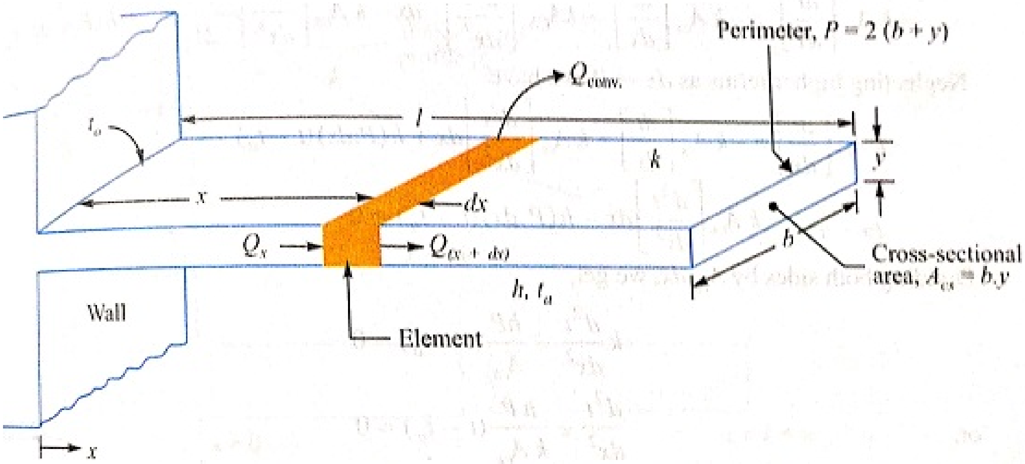


Fig. Rectangular fin of uniform cross-section.

Let, l=length of the fin

b=width of the fin

y=thickness of the fin

P= perimeter of the fin

A_{cs}=area of cross-section(=by)

t_o=temperature at the base of the fin

t_a= temperature of the surrounding fluid

k=thermal conductivity and

h=heat transfer coefficient.

In order to determine the governing differential equation for the fins as shown in the figure above, consider the heat flow to and from an element dx thick at a distance x from the base.

Heat conducted into the element a plane x,

$$Q_x = -kA_{cs}\left[\frac{dT}{dx}\right]_x \quad \text{...(i)}$$

Heat conducted out of the element at plane (x+dx)

$$Q_{(x+dx)} = -kA_{cs}\left[\frac{dT}{dx}\right]_{x+dx} \quad \text{..(ii)}$$

Heat convected out of the element between the planes x and(x+dx)

$$Q_{conv} = h(P.dx)(t - t_a)$$

Applying an energy balance on the element , we cn write

$$Q_x = Q_{(x+dx)} + Q_{conv}$$

$$-kA_{cs}\left[\frac{dT}{dx}\right]_x = -kA_{cs}\left[\frac{dT}{dx}\right]_{x+dx} + h(P.dx)(t-t_a) \quad \text{...(1)}$$

Making a Taylor's expansion of the temperature gradient at (x+dx) in terms of that at x , we get

$$\left(\frac{dt}{dx}\right)_{x+dx} = \left(\frac{dt}{dx}\right)_x + \frac{d}{dx}\left(\frac{dt}{dx}\right)_x dx + \frac{d^2}{dx^2}\left(\frac{dt}{dx}\right)\frac{(dx)^2}{\angle 2} + \text{............}$$

Substituting this in equation(1), we have

$$-kA_{cs}\left[\frac{dT}{dx}\right]_x = -kA_{cs}\left[\frac{dT}{dx}\right]_x - kA_{cs}\left[\frac{d^2t}{dx^2}\right]_x dx - kA_{cs}\left[\frac{d^3t}{dx^3}\right]\frac{(dx)^2}{\angle 2} + \text{.......} + h(P.dx)(t-t_a)$$

Neglecting higher terms as dx $\rightarrow 0$, we have

$$-kA_{cs}\left[\frac{dT}{dx}\right] = -kA_{cs}\left[\frac{dT}{dx}\right] - kA_{cs}\left[\frac{d^2t}{dx^2}\right]dx + h(P.dx)(t-t_a)$$

Or, $$kA_{cs}\left[\frac{d^2t}{dx^2}\right]dx - h(P.dx)(t-t_a) = 0$$

Dividing both sides by A_{cs}dx, we get,

$$k\frac{d^2t}{dx^2}-\frac{hP}{A_{cs}}(t-t_a)=0$$

Or, $$\frac{d^2t}{dx^2}-\frac{hP}{kA_{cs}}(t-t_a)=0 \quad \text{………………………………………(2)}$$

Let us assume $\theta_{(x)}=t_{(x)}-t_{(a)}$

As the ambient temperature is constant we get by differentiation

$$\frac{d\theta}{dx}=\frac{dt}{dx},\frac{d^2\theta}{dx^2}=\frac{d^2t}{dx^2}$$

Thus, $\frac{d^2\theta}{dx^2}-m^2\theta=0$ ………………………………………………………….(3)

Where , m= $\sqrt{\frac{hP}{kA_{cs}}}$

Equations (2) and(3) represent a general form of the energy equation for one dimensional heat dissipation from the extended surface (fin).

The general solution of this linear and homogeneous second order differential equation is the form :

$$\theta=C_1e^{mx}+C_2e^{-mx} \quad \text{……………………………………………………(4)}$$

Or, [t-t_a= $C_1e^{mx}+C_2e^{-mx}$]

Where C_1 and C_2 are the constants, these are to be determined by using proper boundary conditions.

One boundary condition is:

$$\theta=\theta_0=t_0-t_a \text{ at x=0}$$

The other boundary condition depends on the physical situation. The following cases may be considered:

CASE I. The fin is infinitely long and the temperature at the end of the fin is essentially that of the surrounding fluid.

CASE II. The end of the fin is insulated.

CASE III. The fin is of finite length and loses heat by convection.

HEAT DISSIPATION FROM AN INFINITELY LONG FIN ($l \rightarrow \infty$) :

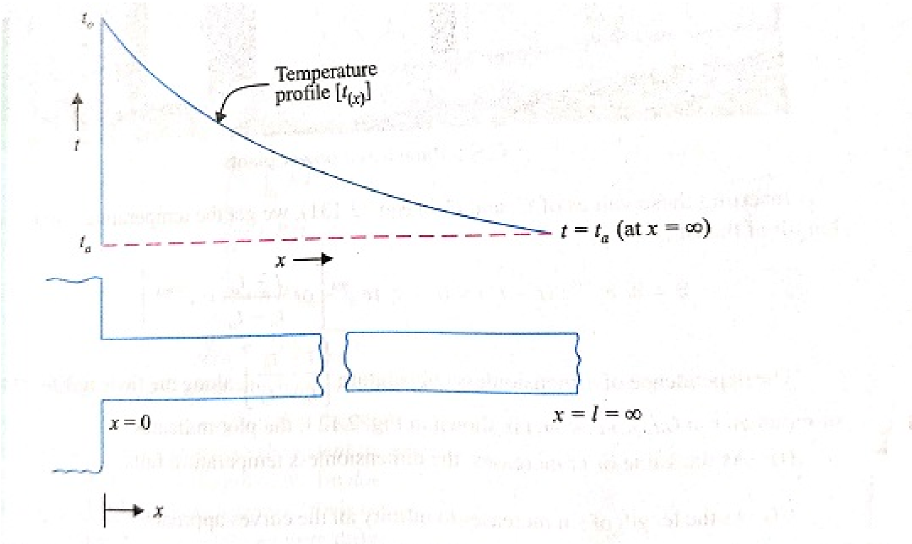


Fig. Infinitely long fin (Case I).

Referring to above figure, the boundary conditions are:

(i) At $x=0$ $t=t_0$ (temperature at the base of fin equals the temperature of the surface to which fin is attached.)

$t-t_a=t_0-t_a$

or, at $x=0$ $\theta=\theta_0$

(ii) At $x=\infty$ $t=t_a$

(temperature at the end of an infinitely long fin equals that of the surroundings)

At $x=\infty$, $\theta=0$

Substituting these boundary conditions in equation-(4), we get

$$C_1 + C_2 = \theta_0 \quad \text{..........(i)}$$

$$C_1 e^{m(\infty)} + C_2 e^{-m(\infty)} = 0 \quad \text{..........(ii)}$$

Or, $C_1 e^{m(\infty)} + 0 = 0$

Hence $C_1=0$

And $C_2=\theta_0$

Inserting these values of C_1 and C_2 in equation-(4), we get the temperature distribution along the length of the fin,

$$\theta = \theta_0 e^{-mx}$$

GENERAL FIN EQUATION-LINK BELOW

https://www.youtube.com/watch?v=7rchIQ_TFRA

Or, $(t - t_a) = (t_0 - t_a)e^{-mx}\left[\frac{t - t_a}{t_0 - t_a} = e^{-mx}\right]$ ....(5)

The heat flow rate can be determined in either of the two ways:

(a) By considering the heat flow across the root(or base) by conduction:

(b) By considering the heat which is transmitted by convection from the surface of the fin to the surrounding fluid.

(a) The rate of heat flow across the vase of the fin is given by (Fourier's equation)

$$Q_{fin} = -kA_{cs}\left[\frac{dt}{dx}\right]_{x=0}$$

$$\left[\frac{dt}{dx}\right]_{x=0} = [-m(t_0 - t_a)e^{-mx}]_{x=0} = -m(t_0 - t_a)$$

Henace, $$Q_{fin} = -kA_{cs}x[-m(t_0 - t_a)] = kA_{cs}m(t_0 - t_a)$$

I,e. $$Q_{fin} = kA_{cs}m(t_0 - t_a)$$

Or, $$Q_{fin} = kA_{cs}\sqrt{\frac{Ph}{kA_{cs}}}(t_0 - t_a)$$

Or, $$Q_{fin} = \sqrt{PhkA_{cs}}(t_0 - t_a) \text{ ....(6)}$$

(b) Alternatively :

$$Q_{fin} = \int_0^\infty h(Pdx)(t - t_a)$$

$$= \int_0^\infty hP(t_0 - t_a)e^{-mx}dx$$

$$= hP(t_0 - t_a)\int_0^\infty e^{-mx}dx$$

$$= hP(t_0 - t_a)\frac{1}{m} = hP(t_0 - t_a)\sqrt{\frac{kA_{cs}}{Ph}}$$

Or, $$Q_{fin} = \sqrt{PhkA_{cs}}(t_0 - t_a)$$

HEAT DISSIPATION FROM AN INFINITELY LONG FIN (l $\to \infty$) :-LINK BELOW

https://www.youtube.com/watch?v=SgEHJghKXF0

NOTE- An infinitely long fin is one for which m$_l \to \infty$ and this condition may be approached when ml>5]

HEAT DISSIPATION FROM A FIN INSULATED AT THE TIP

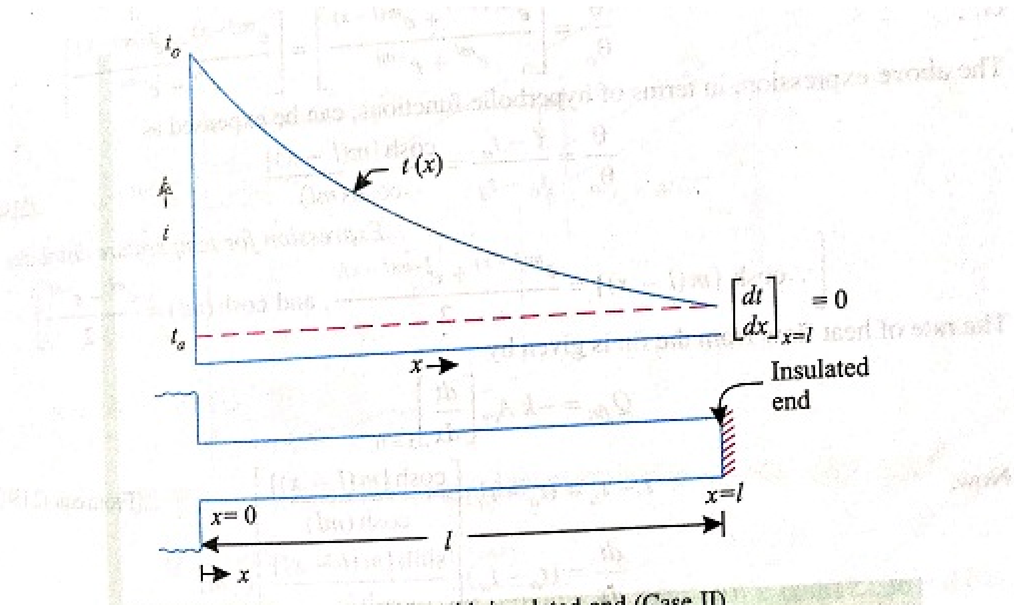


Fig. The fin with insulated end (Case II).

The boundary conditions are:

(i) At x=0, $\theta=\theta_0$

(ii) At x=l, dt/dx=0

Applying these boundary conditions, we have

$C_1+C_2=\theta_0$ ....(i)

Further $$t - t_a = C_1 e^{mx} + C_2 e^{-mx}$$

$$\frac{dt}{dx} = mC_1 e^{mx} - mC_2 e^{-mx}$$

$$\left[\frac{dt}{dx}\right]_{x=l} = mC_1 e^{ml} - mC_2 e^{-ml} = 0$$

Hence, $C_1 e^{ml} - C_2 e^{-ml} = 0$ ....(ii)

Solving equation-(i) and (ii), we have

$$C_2 = \theta_0 - C_1$$

$$C_1 e^{ml} - (\theta_0 - C_1)e^{-ml} = 0$$

Or, $C_1 e^{ml} - \theta_0 e^{-ml} + C_1 e^{-ml} = 0$

Or, $C_1 (e^{ml} + e^{-ml}) = \theta_0 e^{-ml}$

Or, $$C_1 = \theta_0 \left[\frac{e^{-ml}}{e^{ml} + e^{-ml}}\right]$$

Hence, $$C_2 = \theta_0 - \left[\theta_0 \left(\frac{e^{-ml}}{e^{ml} + e^{-ml}}\right)\right]$$

Or, $$C_2 = \theta_0 \left[1 - \frac{e^{-ml}}{e^{ml} + e^{-ml}}\right] = \theta_0 \left[\frac{e^{ml}}{e^{ml} + e^{-ml}}\right]$$

Inserting the value in equation (4),we have

$$\theta = \theta_0 \left[\frac{e^{-ml}}{e^{ml} + e^{-ml}}\right] e^{mx} + \theta_0 \left[\frac{e^{ml}}{e^{ml} + e^{-ml}}\right] e^{-mx}$$

Or, $$\frac{\theta}{\theta_0} = \left[\frac{e^{m(x-l)} + e^{m(l-x)}}{e^{ml} + e^{-ml}}\right] = \left[\frac{e^{m(l-x)} + e^{[-m(l-x)]}}{e^{ml} + e^{-ml}}\right]$$

The above expression , in terms of hyperbolic functions can be expressed as

$$\frac{\theta}{\theta_0} = \frac{t - t_a}{t_0 - t_a} = \frac{\cosh[m(l-x)]}{\cosh(ml)}$$ .....expression for temperature distribution

HEAT DISSIPATION FROM A FIN INSULATED AT THE TIP –LINK BELOW

https://www.youtube.com/watch?v=CpiPTOGT3tY

The rate of heat flow from the fin is given by

$$Q_{fin} = -kA_{cs}\left[\frac{dt}{dx}\right]_{x=o}$$

Now, $$t - t_a = (t_0 - t_a)\left[\frac{\cosh[m(l-x)]}{\cosh(ml)}\right]$$

$$\frac{dt}{dx} = (t_0 - t_a)\left[\frac{\sinh[m(l-x)]}{\cosh(ml)}\right](-m)$$

$$[\frac{dt}{dx}]_{x=0} = -m(t_0 - t_a)\tanh(ml)$$

Hence, $$Q_{fin} = kA_{cs}m(t_0 - t_a)\tanh(ml)$$

$$Q_{fin} = \sqrt{PhkA_{cs}}\,(t_0 - t_a)\tanh(ml)$$

HEAT DISSIPATION FROM A FIN LOSING HEAT AT THE TIP

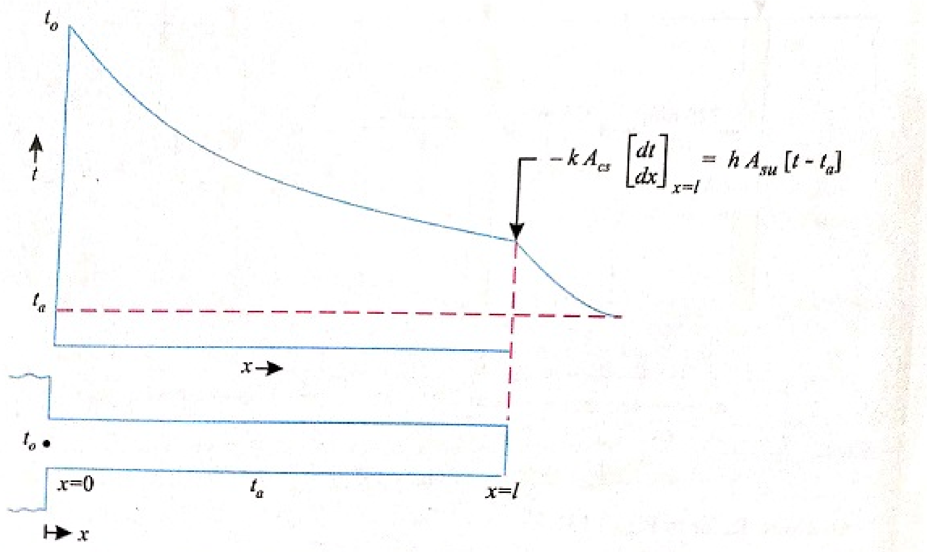


Fig. . A fin of finite length losing heat at tip (case III).

The boundary conditions are:

(i) At x=0 , $\theta=\theta_0$

(ii) Heat conducted to the fin at x=l= Heat convected from the end to the surroundings.

I.e., $$-kA_{cs}\left[\frac{dt}{dx}\right]_{x=l} = hA_{su}(t-t_a)$$

Where A_{cs} (cross sectional area for heat conduction) equals A_{su} (surface area from which the convective heat transport takes place), at the tip of the fin; $A_{cs}= A_{su}$.

Thus , $$\frac{dt}{dx} = -\frac{h\theta}{k} \quad \text{at } x=l$$

Applying these boundary conditions we get,

$$C_1 + C_2 = \theta_0 \quad \text{..(I)}$$

Further $$t - t_a = C_1 e^{mx} + C_2 e^{-mx}$$

Differentiating this expression with respect to x, we have

$$\frac{dt}{dx} = mC_1 e^{mx} - mC_2 e^{-mx}$$

$$\left[\frac{dt}{dx}\right]_{x=l} = mC_1 e^{ml} - mC_2 e^{-ml} = -\frac{h\theta}{k}$$

Or, $$C_1 e^{ml} - C_2 e^{-ml} = -\frac{h\theta}{km}$$

Or, $$C_1 e^{ml} - C_2 e^{-ml} = -\frac{h}{km}[C_1 e^{ml} + C_2 e^{-ml}] \quad \text{..(ii)}$$

Solving equation (i) and (ii) we have,

$$C_2 = \theta_0 - C_1 \quad \text{......................from equation-(i)}$$

$$C_1 e^{ml} - (\theta_0 - C_1)e^{-ml} = -\frac{h}{km}[C_1 e^{ml} + (\theta_0 - C_1)e^{-ml}] \quad \text{...............................from equation-(ii)}$$

$$C_1 e^{ml} - \theta_0 e^{-ml} + C_1 e^{ml} = -\frac{h}{km}C_1 e^{ml} - \frac{h}{km}\theta_0 e^{-ml} + \frac{h}{km}C_1 e^{-ml}$$

$$C_1[(e^{ml} + e^{-ml)} + \frac{h}{km}e^{ml} - \frac{h}{km}e^{-ml}] = \theta_0 e^{-ml} - \frac{h}{km}\theta_0 e^{-ml}$$

$$C_1[(e^{ml}+e^{-ml})+\frac{h}{km}(e^{ml}-e^{-ml})]=\theta_0 e^{-ml}[1-\frac{h}{km}]$$

Hence, $$C_1=\frac{\theta_0\left[1-\frac{h}{km}\right]e^{-ml}}{\left[(e^{ml}+e^{-ml}\right]+\frac{h}{km}(e^{ml}-e^{-ml})]}$$

And, $$C_2=\theta_0-\left[\frac{\theta_0\left[1-\frac{h}{km}\right]e^{-ml}}{\left[(e^{ml}+e^{-ml}\right]+\frac{h}{km}(e^{ml}-e^{-ml})]}\right]$$

$$=\theta_0\left[1-\frac{(1-\frac{h}{km})e^{-ml}}{(e^{ml}+e^{-ml})+\frac{h}{km}(e^{ml}-e^{-ml})}\right]$$

$$=\theta_0\left[\frac{(e^{ml}+e^{-ml})+\frac{h}{km}(e^{ml}-e^{-ml})-e^{-ml}+\frac{h}{km}e^{-ml}}{(e^{ml}+e^{-ml})+\frac{h}{km}(e^{ml}-e^{-ml})}\right]$$

$$=\theta_0\left[\frac{e^{ml}+e^{-ml}+\frac{h}{km}e^{ml}-\frac{h}{km}e^{-ml}-e^{-ml}+\frac{h}{km}e^{-ml}}{(e^{ml}+e^{-ml})+\frac{h}{km}(e^{ml}-e^{-ml})}\right]$$

Or, $$C_2=\frac{\theta_0\left[1+\frac{h}{km}\right]e^{ml}}{\left[(e^{ml}+e^{-ml})+\frac{h}{km}(e^{ml}-e^{-ml})\right]}$$

Substituting these values of constants C_1 and C_2 , we get

$$\theta=C_1e^{mx}+C_2e^{-mx}$$

$$\theta=\left[\frac{\theta_0(1-\frac{h}{km})e^{-ml}}{(e^{ml}+e^{-ml})+\frac{h}{km}(e^{ml}-e^{-ml})}\right]e^{mx}+\left[\frac{\theta_0(1+\frac{h}{km})e^{ml}}{(e^{ml}+e^{-ml})+\frac{h}{km}(e^{ml}-e^{-ml})}\right]e^{-mx}$$

Or, $$\frac{\theta}{\theta_0}=\frac{\left[e^{m(l-x)}+e^{-m(l-x)}\right]+\frac{h}{km}[e^{m(l-x)}+e^{-m(l-x)}]}{[(e^{ml}+e^{-ml})+\frac{h}{km}(e^{ml}-e^{-ml})]}$$

Or, $$\frac{\theta}{\theta_0}=\frac{t-t_a}{t_0-t_a}=\frac{\cosh[m(l-x)]+\frac{h}{km}[\sinh\{m(l-x)\}]}{\cosh(ml)+\frac{h}{km}[\sinh(ml)]}$$

The rate of heat flow from the fin is given by

$$Q_{fin}=-kA_{cs}\left[\frac{dt}{dx}\right]_{x=0}$$

Now $$t-t_a=(t_0-t_a)\left[\frac{\cosh[m(l-x)]+\frac{h}{km}[\sinh\{m(l-x)\}]}{\cosh(ml)+\frac{h}{km}[\sinh(ml)]}\right]$$

Differentiating the above expression with respect to x , we get

$$\frac{dt}{dx}=(t_0-t_a)\left[\frac{-m\sinh\{m(l-x)\}-m[\frac{h}{km}\{\cosh[m(l-x)]\}}{\cosh(ml)+\frac{h}{km}[\sinh(ml)]}\right]$$

$$\left[\frac{dt}{dx}\right]_{x=0}=-(t_0-t_a)m\left[\frac{\sinh(ml)+\frac{h}{km}\{\cosh(ml)\}}{\cosh(ml)+\frac{h}{km}\{\sinh(ml)\}}\right]$$

Hence, $$Q_{fin}=kA_{cs}m(t_0-t_a)\left[\frac{\sinh(ml)+\frac{h}{km}\{\cosh(ml)\}}{\cosh(ml)+\frac{h}{km}\{\sinh(ml)\}}\right]$$

$$=\sqrt{PhkA_{cs}}(t_0-t_a)\left[\frac{\sinh(ml)+\frac{h}{km}\{\cosh(ml)\}}{\cosh(ml)+\frac{h}{km}\{\sinh(ml)\}}\right]$$

$$Q_{fin} = \sqrt{PhkA_{cs}}(t_0 - t_a)\left[\frac{\tanh(ml) + \frac{h}{km}}{1 + \frac{h}{km}\tanh(ml)}\right]$$

Temperature distribution & heat transfer rate for fins of uniform cross sectional area

case	Tip condition(x= l)	Temperature distribution	Fin heat transfer rate (Q_{fin})
A	Convection heat transfer	$\frac{\theta}{\theta_0} = \frac{t - t_a}{t_0 - t_a} = \frac{\cosh[m(l-x)] + \frac{h}{km}[\sinh\{m(l-x)\}]}{\cosh(ml) + \frac{h}{km}[\sinh(ml)]}$	$Q_{fin} = \sqrt{PhkA_{cs}}(t_0 - t_a)\left[\frac{\tanh(ml) + \frac{h}{km}}{1 + \frac{h}{km}\tanh(ml)}\right]$
B	Insulated tip	$\frac{\theta}{\theta_0} = \frac{t - t_a}{t_0 - t_a} = \frac{\cosh[m(l-x)]}{\cosh(ml)}$	$Q_{fin} = \sqrt{PhkA_{cs}}(t_0 - t_a)\tanh(ml)$
C	Infinite Fin Length	$\theta = \theta_0 e^{-mx}$ Or $(t - t_a) = (t_0 - t_a)e^{-mx}\left[\frac{t - t_a}{t_0 - t_a} = e^{-mx}\right]$	$Q_{fin} = \sqrt{PhkA_{cs}}(t_0 - t_a)$

EFFICIENCY AND EFFECTIVENESS OF FIN

Efficiency of fin(η_{fin}):

The efficiency of a fin is defined as the ratio of the actual heat transferred by the fin to the maximum heat transferrable by fin, if entire fin area were at base temperature.

i.e.

η_{fin}=Actual heat transferred by the fin(Q_{fin})/Maximum heat that would be transferred if whole surface of the fin is maintained at the base temperature(Q_{max})

For a fin which is infinitely long

$$\eta_{fin} = \frac{\sqrt{PhkA_{cs}}(t_0 - t_a)}{hPl(t_0 - t_a)} = \sqrt{\frac{kA_{cs}}{hPl^2}} = \frac{1}{ml}$$

For a fin which is insulated at the tip

$$\eta_{fin} = \frac{\sqrt{PhkA_{cs}}(t_0 - t_a)\tanh(ml)}{hPl(t_0 - t_a)} = \sqrt{\frac{kA_{cs}}{hPl^2}} = \frac{\tanh(ml)}{ml}$$

Effectiveness of fin(ϵ_{fin})

Effectiveness of the fin is the ratio of the fin heat transfer rate to the heat transfer rate that would exist without a fin.

$$\in_{fin} = \frac{Q_{withfin}}{Q_{wiyhoutfin}} = \frac{\sqrt{PhkA_{cs}}(t_0 - t_a)}{hA_{cs}(t_0 - t_a)} = \sqrt{\frac{Pk}{hA_{cs}}}$$

- Fin effectiveness $\sqrt{\frac{Pk}{hA_{cs}}}$ should be greater than unity if the rate of heat transfer from the primary surface is to be improved. It has been observed that use of fins on surfaces is justified only if $\frac{Pk}{hA_{cs}}$ >5.
- If the ratio of P and A_{cs} is increased the effectiveness of fin is improved. Due to this reason , thin and closely spaced fins are preferred.
- Use of fins is only justified where h is small(h<0.25$\left[\frac{kP}{A}\right]$.
- The use of fins will be more effective with materials of large thermal conductivities[Although copper is superior to aluminium regarding thermal conductivity, yet fins are generally made of aluminium since it is cheaper and lighter in weight.]

Relation between η_{fin} and ϵ_{fin} :

$$\eta_{fin} = \frac{\sqrt{PhkA_{cs}}(t_0 - t_a)\tanh(ml)}{hPl(t_0 - t_a)} \quad \text{.....................................(i)}$$

$$\in_{fin} = \frac{\sqrt{PhkA_{cs}}(t_0 - t_a)\tanh(ml)}{hA_{cs}(t_0 - t_a)} \quad \text{...(ii)}$$

Dividing equation-(ii) by (i), we have

$$\frac{\in_{fin}}{\eta_{fin}} = \frac{Pl}{A_{cs}}$$

Or, $$\in_{fin} = \eta_{fin}\frac{Pl}{A_{cs}} = \eta_{fin} x \frac{surfaceareaoffin}{cross - \sec tionalareaoffin}$$

<u>EFFICIENCY AND EFFECTIVENESS OF FIN-LINK BELOW</u>

https://www.youtube.com/watch?v=ywGdNKTK6a8

<u>CONDUCTION-UNSTEADY –STATE (TRANSIENT)</u>

- If the temperature of a body does not vary with time, it is said to be in a steady state. But if there is an abrupt change in its surface temperature , it attains an equilibrium temperature or a steady state after some period.
- During this period the temperature varies with time and the body is said to be in an unsteady or transient state.
- The term transient or unsteady designates a phenomenon which is time dependent.
- Transient conditions occur in :

(i) Cooling of I.C. engines.,

(ii) Automobile engines,

(iii) Heating and cooling of metal billets,

(iv) Cooling and freezing of foods,

(v) Heat treatment of metals by quenching,

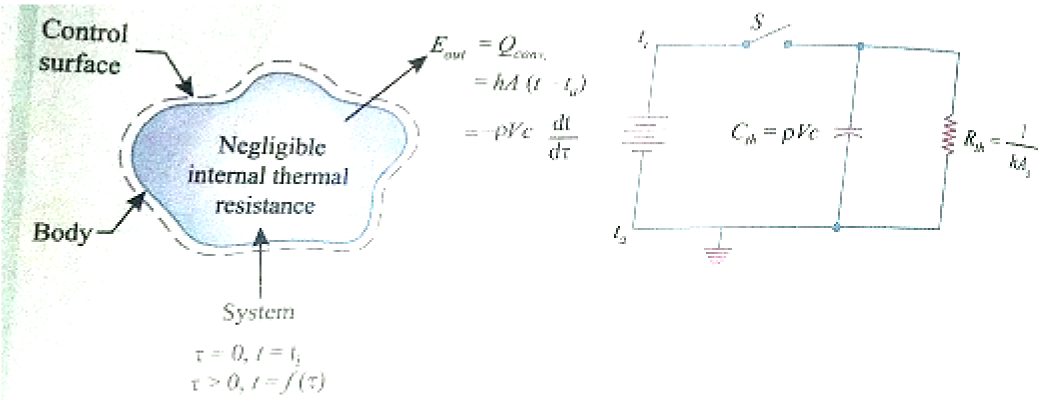
(vi) Brick burning. Etc.

The temperature field in any transient problem, is given by

$$T=f(x,y,z,\tau)$$

HEAT CONDUCTION IN SOLIDS HAVING INFINITE THERMAL CONDUCTIVITY (NEGLIGIBLE INTERNAL RESISTANCE)-LUMPED PARAMETER ANALYSIS

- All solids have a finite thermal conductivity and there will be always a temperature gradient inside the solid whenever heat is added or removed. However , for solids of large thermal conductivity with surface areas that are large in proportion to their volume like plates and thin metallic wires, the internal resistance (L/KA) can be assumed to be small or negligible in comparison with the convective resistance(1/hA) at the surface. Typical examples is heat treatment of metals.
- The process in which the internal resistance is assumed negligible in comparison with its surface resistance is called the Newtonian heating or cooling process. The temperature is considered to be uniform at a given time. Such an analysis is called Lumped parameter analysis because the whole solid , whose energy at any time is a function of its temperature and total heat capacity is treated as one lump.



(*a*) General system for unsteady heat conduction

(*b*) Equivalent thermal circuit for lumped capacitance solid

Fig. . Lumped heat capacity system.

- Let us consider a body whose initial temperature is t_i throughout and which is placed suddenly in ambient air or any liquid at a constant temperature t_a as shown in the figure (a) .
- The transient response of the body can be determined by relating its rate of change of internal energy with convective exchange at the surface. That is:

$$Q = -\rho VC \frac{dt}{d\tau} = hA_s (t - t_a) \qquad \text{..........................(1)}$$

Where, ρ = density of solid

V= volume of the body

C= specific heat of body

h= convective heat transfer coefficient

t= temperature of body at any time

A_s =surface area of the body

t_a = ambient temperature

τ =time

After arranging equation (1) and integrating we get

$$\int \frac{dt}{(t-t_a)} = -\frac{hA_s}{\rho Vc}\int d\tau \text{ ....(2)}$$

Or, $$\ln(t-t_a) = -\frac{hA_s}{\rho Vc}\tau + C_1 \text{ ....(3)}$$

The boundary conditions are:

At τ=0, $t=t_i$ (initial surface temperature)

Hence , $C_1 = \ln(t_i - t_a)$

Hence, $$\ln(t-t_a) = -\frac{hA_s}{\rho Vc}\tau + \ln(t_i - t_a)$$

Or, $$\frac{t-t_a}{t_i-t_a} = \frac{\theta}{\theta i} = \exp\left[-\frac{hA_s}{\rho Vc}\tau\right] \text{ ....(4)}$$

Following points are to be noted,

1. Equation (4) gives the temperature distribution in the body for Newtonian heating or cooling and it indicates that temperature rises exponentially with time as shown in figure below.

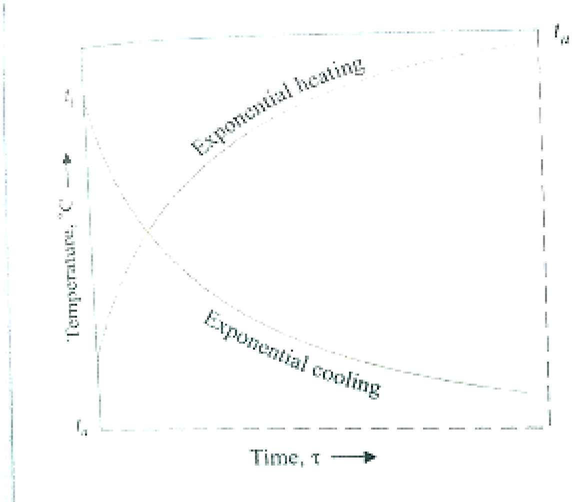


Fig. Newtonian heating or cooling.

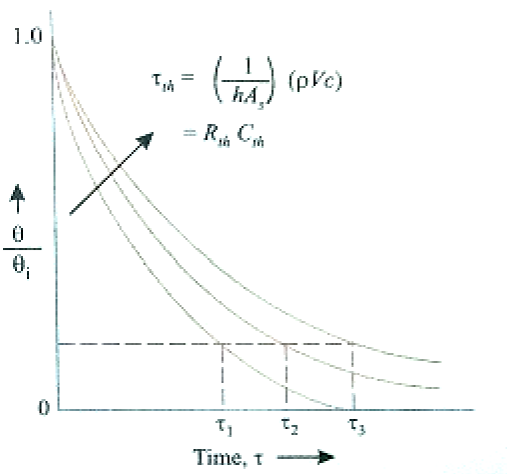


Fig. Transient temperature response.

2. The quantity $\rho Vc/hA_s$ has the dimensions of time and is called thermal time constant, denoted by τ_{th} . Its value is indicative of the rate of response of a system to a sudden change in its environmental temperature.

$$\tau_{th} = \left(\frac{1}{hA_s}\right)(\rho Vc) = R_{th}C_{th}$$

where, $$R_{th} = \left(\frac{1}{hA_s}\right) = responsetoconvectionheattransfer$$

$$C_{th}(= \rho Vc) = lumpedthermalcapaci\tan ceofsolid$$

Figure above shows that any increase in R_{th} or C_{th} will cause a solid to respond more slowly to changes in its thermal environment and will increase the time required to attain the thermal equilibrium (θ=0)

Figure (b) shows an analogus electric network for a lumped heat capcity system, in which C_{th}=ρVc represents the thermal capacity of the system. The value of C_{th} can be obtained from the following thermal and electrical equations , by similarity.

$$Q = (\rho Vc)t = C_{th}.t$$ ……………………….thermal equation.

s =C.E ……………………………………………electrical equation.

Where, s= capacitor charge

C = capacitance of the condenser, and

E = voltage.

When the switch is closed the solid is charged to the temperature θ. On opening the switch , the thermal energy stored as C_{th} is dissipated through the thermal resistance $R_{th}=(1/hA_s)$ and the temperature of the body decays with time. From this analogy it is concluded that RC electrical circuits may be used to determine the transient behavior of thermal systems.

The power on exponential, i.e. $(hA_s/\rho Vc)\tau$ can be arranged in dimensionless form as follows.

$$\frac{hA_s}{\rho Vc}\tau = \left(\frac{hV}{kA_s}\right)\left(\frac{A_s^2 k}{\rho V^2 c}\tau\right) = \left(\frac{hL_c}{k}\right)\left(\frac{\alpha\tau}{L_c^2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Where $\alpha = \left[\frac{k}{\rho c}\right]$= thermal diffusivity of solid

L_c =characteristic length= volume of the solid (V)/surface area of the solid (A_s)

The values of L_c , for simple geometric shapes , are given below

Flat plate: $L_c = \frac{V}{A_s} = \frac{LBH}{2BH} = L/2 = semithickness$

Where, L, B and H are thickness, width and eight of the plate

Cylinder: $L_c = \frac{\pi R^2 L}{2\pi RL} = \frac{R}{2}$ where R is the radius of the cylinder.

Sphere: $L_c = \frac{\frac{4}{3}\pi R^3}{4\pi R^2} = \frac{R}{3}$ where R is the radius of the sphere.

Cube: $L_c = \frac{L^3}{6L^2} = \frac{L}{6}$ where L is the side of the cube.

From equation (5)

i. the non dimensional factor hL_c/k is called the Biot number B_i,

i.e. $Bi = \frac{hL_c}{k} = Biotnumber.$

It gives an indication of the ratio of internal (conduction) resistance to surface (convection) resistance. When the value of Bi is small, it indicates that the system has a small internal (conduction) resistance , i.e. relatively small temperature gradient or the existence of practically uniform temperature within the system . The convective resistance predominates and the transient phenomenon is controlled by the convective heat exchange.

If Bi<0.1 , the lumped heat capacity approach can be used to advantage with simple shapes such as plates, cylinders, spheres and cubes.

ii. The non dimensional factor $\frac{\alpha\tau}{L_c^2}$ is called the Fourier number, F_O.

i.e. $$F_o = \frac{\alpha\tau}{L_c^2}$$

It signifies the degree of penetration of heating or cooling effect through a solid.

Using non dimensional terms , equation (4) takes the form

$$\frac{\theta}{\theta_i} = \frac{t - t_a}{t_0 - t_a} = e^{-BiF_o} \quad \text{..(6)}$$

IANSTATANEOUS HEAT FLOW RATE AND TOTAL HEAT TRANSFER:

The instantaneous rate of heat flow (Q_i) may be found as follows:

$$Q_i = \rho Vc \frac{dt}{d\tau} = \rho Vc \frac{d}{d\tau}\left[t_a + (t_i - t_a)\exp\left\{ -\frac{hA_s}{\rho Vc}\tau \right\} \right]$$

$$Q_i = \rho Vc \left[(t_i - t_a)\left\{ -\frac{hA_s}{\rho Vc} \right\} \exp\left\{ -\frac{hA_s}{\rho Vc}\tau \right\} \right]$$

$$Q_i = -hA_s(t_i - t_a)\exp\left[-\frac{hA_s}{\rho Vc}\tau \right] \quad \text{..(7)}$$

$$Q_i = -hA_s(t_i - t_a)e^{-BiF_o} \quad \text{...(7(a))}$$

The total heat transfer is

$$Q' = \int_0^\tau Q_i d\tau$$

$$= \int_0^\tau -hA_s(t_i - t_a)\exp\left[-\frac{hA_s}{\rho Vc}\tau \right] d\tau$$

$$= \left[-hA_s(t_i - t_a)\frac{\exp(-hA_s/\rho Vc)\tau}{-hA_s/\rho Vc} \right]_0^\tau$$

$$= \rho Vc(t_i - t_a)\left[\exp\left\{-\frac{hA_s}{\rho Vc}\tau\right\}\right]_0^{\tau}$$

$$Q' = \rho Vc(t_i - t_a)\left[\exp\left\{-\frac{hA_s}{\rho Vc}\right\} - 1\right]$$

$Q' = \rho Vc(t_i - t_a)\left[e^{-BiF_o} - 1\right]$..............................in terms of non dimensional Bi and F_o number.

LUMPED PARAMETER ANALYSIS-LINK BELOW

https://www.youtube.com/watch?v=mXC-2LVZB34

REFERENCES

1. Heat and Mass transfer, author-R.K.RAJPUT
2. HEAT AND MASS TRANSFER, AUTHOR-FRANK P. INCROPERA , DAVID P. DEWITT
3. GATE STUDY MATERIAL, ACE ENGINEERING ACADEMY
4. GOOGLE IMAGE
5. YOUTUBE